Examen #2 (Soluciones). Lunes 10 de junio de 2013

Cálculo Integral UAM-Azcapotzalco

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(1) Calcule $\int \sec x \tan x \, dx$.

Sol (A) De Cálculo Diferencial, sabemos que:
Solución A

$$\frac{d}{dx}\boxed{\sec x} = \sec x \tan x.$$

Por la definición de anti-derivada, tenemos que:

$$\int \sec x \tan x \, dx = \boxed{\sec x} + C.$$

Sol (B) Tenemos que:
Solución B.

$$F(x) = \int_a^x \sec t \tan t \, dt.$$

es una función de x. De Cálculo Diferencial:
sabemos que:

$$F(x) = \int_a^x \frac{d}{dt}(\sec t)\, dt.$$

Por el Teorema Fundamental del Cálculo se sigue que

$$F(x) = \int_a^x \frac{d}{dt}(\sec t)\, dt = \sec x - \sec a$$

i.e.

$$F(x) = \boxed{\int \sec x \tan x \, dx = \sec x + C}$$

(2.) Calcule $\displaystyle\int_0^{2\pi} \sqrt{\frac{1-\cos x}{2}}\, dx$

Como $\sin^2 \theta = \dfrac{1-\cos 2\theta}{2}$

entonces

$$\sin^2 \frac{x}{2} = \frac{1-\cos x}{2}$$

Así:

$$\int_0^{2\pi} \sqrt{\frac{1-\cos x}{2}}\, dx = \int_0^{2\pi} \sqrt{\sin^2 \frac{x}{2}}\, dx$$

$$= \int_0^{2\pi} \left|\sin \frac{x}{2}\right| dx$$

Si $x = 2\varphi$, $\Rightarrow$ $\left.\begin{matrix} x=0 \\ x=2\pi \end{matrix}\right\} \Rightarrow \begin{matrix} \varphi = 0 \\ \varphi = \pi. \end{matrix}$ y $dx = 2d\varphi$

$$= \int_0^{\pi} |\sin\varphi|\, 2\, d\varphi = 2\int_0^{\pi} \sin\varphi\, d\varphi,$$ pues $\sin\varphi \geq 0$ en $\varphi \in [0,\pi]$

$$= -2\cos\varphi\Big|_0^{\pi} = -2\left\{\cos\pi - \cos 0\right\} = -2[-1-1]$$

$$= 4 \quad \Rightarrow \quad \boxed{\int_0^{2\pi} \sqrt{\frac{1-\cos x}{2}}\, dx = 4}$$

(3) Calcule $\displaystyle\int \frac{8}{(4x^2+1)^2}\,dx$

Sea $2x = \tan\theta$. Entonces: $\dfrac{dx}{d\theta} = \dfrac{1}{2}\left(1+\tan^2\theta\right)$

$$dx = \frac{1}{2}\left(1+\tan^2\theta\right)d\theta.$$

Así:

$$4x^2+1 = \tan^2\theta + 1.$$

Entonces:

$$\int \frac{8}{(4x^2+1)^2}\,dx = \int \frac{8}{(\tan^2\theta+1)^2}\cdot\frac{1}{2}\left(1+\tan^2\theta\right)d\theta$$

$$= 4\int \frac{d\theta}{1+\tan^2\theta} = 4\int \frac{d\theta}{\sec^2\theta} = 4\int \cos^2\theta\, d\theta$$

$$= 4\int \frac{1+\cos 2\theta}{2}\,d\theta = 2\int (1+\cos 2\theta)\,d\theta$$

$$= 2\left[\int d\theta + \int \cos 2\theta\, d\theta\right] = 2\left[\theta + \frac{1}{2}\sin 2\theta\right] + C$$

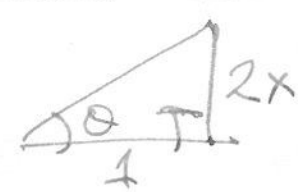
$$= 2\theta + \sin(2\theta) + C \qquad = 2\theta + \sin(2\theta) + C$$

Puesto que $2x = \tan\theta \Rightarrow \theta = \operatorname{Arctan}(2x)$.

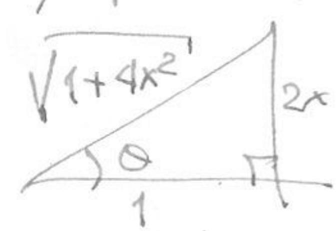
Por otra parte:

$$\sin 2\theta = 2\sin\theta\cos\theta$$

Ahora, de la definición $\tan\theta = 2x$, podemos dibujar el triángulo rectángulo:



$\Rightarrow$



Entonces:

$$\sin\theta = \frac{2x}{\sqrt{4x^2+1}} \quad y \quad \cos x = \frac{1}{\sqrt{4x^2+1}}$$

Así

$$\sin(2\theta) = 2\sin\theta\cos\theta = 2 \cdot \frac{2x}{\sqrt{4x^2+1}} \cdot \frac{1}{\sqrt{4x^2+1}}$$

$$= \frac{4x}{1+4x^2}.$$

Por lo tanto

$$\int \frac{8}{(4x^2+1)^2}\,dx = 2\theta + \sin 2\theta + C$$

i.e.

$$\boxed{\int \frac{8}{(4x^2+1)^2}\,dx = 2\operatorname{Arctan} 2x + \frac{4x}{1+4x^2} + C}$$

(4) Calcule $\int \frac{1}{x^4-16}\,dx$

Fracciones Parciales:

$$\frac{1}{x^4-16} = \frac{1}{(x^2-4)(x^2+4)} = \frac{1}{(x-2)(x+2)(x^2+4)} =$$

$$= \frac{A}{x-2} + \frac{B}{x+2} + \frac{Cx+D}{x^2+4}.$$

Entonces

$$\frac{1}{x^4-16} = \frac{A(x+2)(x^2+4) + B(x-2)(x^2+4) + (Cx+D)(x-2)(x+2)}{(x-2)(x+2)(x^2+4)}.$$

Entonces.

$$1 = A(x+2)(x^2+4) + B(x-2)(x^2+4) + (Cx+D)(x-2)(x+2).$$

Tomo $\underline{x=2}$

$$1 = A(2+2)(4+4) + 0 + 0$$

$$\Rightarrow \quad 1 = 32A \quad \Rightarrow \quad \boxed{A = \frac{1}{32}}$$

$\underline{x=-2}$

$$1 = 0 + B(-2-2)(4+4) + 0$$

$$1 = -32B \quad \Rightarrow \quad \boxed{B = \frac{-1}{32}}$$

$\underline{x=0}$

$$1 = A(2)(4) + B(-2)(4) + (C\cdot 0+D)(-2)(2)$$

$$1 = 8A - 8B - 4D.$$

$$1 = \frac{8}{32} + \frac{8}{32} - 4D$$

$$\Rightarrow \quad \frac{16}{32} - 4D = 1$$

$$-4D = 1 - \frac{16}{32} = 1$$

$$-4D = \frac{16}{32} = \frac{1}{2}$$

$$\Rightarrow \boxed{D = -\frac{1}{8}}$$

$\boxed{x = 1}$

$$1 = A(3)(5) + B(-1)(5) + (C+D)(-1)(3)$$

$$15A - 5B - 3(C+D) = 1.$$

Sustituir los valores de A, B y D:

$$\frac{15}{32} + \frac{5}{32} - 3\left(C - \frac{1}{8}\right) = 1$$

$$\frac{20}{32} - 3C + \frac{3}{8} = 1$$

$$\frac{20}{32} + \frac{12}{32} - 3C = 1$$

$$\frac{32}{32} - 3C = 1$$

$$1 - 3C = 1$$

$$-3C = 0 \Rightarrow \boxed{C = 0}$$

Entonces:

$$\frac{1}{x^4-16} = \frac{1}{32}\frac{1}{x-2} - \frac{1}{32}\frac{1}{x+2} + \left(\frac{-1}{8}\right)\frac{1}{x^2+4}$$

Integrando

$$\int \frac{dx}{x^4-16} = \frac{1}{32}\operatorname{Log}|x-2| - \frac{1}{32}\operatorname{Log}|x+2| - \frac{1}{8\cdot 2}\operatorname{Arctan}\left(\frac{x}{2}\right) + C.$$

i.e.

$$\left(\int \operatorname{Arctan}\left(\int \frac{1}{x^2+A^2}\,dx = \frac{1}{|A|}\operatorname{Arctan}\left(\frac{x}{|A|}\right)\right).\right)$$

i.e. $\boxed{\int \frac{dx}{x^4-16} = \frac{1}{32}\operatorname{Log}\left|\frac{x-2}{x+2}\right| - \frac{1}{16}\operatorname{Arctan}\frac{x}{2} + C}$

(5) Calcule $\int_0^\infty \frac{1}{(x+1)\sqrt{x}}\,dx$.

Calculamos primero la integral. Sea $y=\sqrt{x}$, $y^2=x$.

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} \Downarrow 2dy = \frac{dx}{\sqrt{x}}.$$

Entonces

$$\int \frac{dx}{(x+1)\sqrt{x}} = \int \frac{1}{y^2+1}\cdot 2dy = 2\operatorname{Arctan} y$$

$$= 2\operatorname{Arctan}(\sqrt{x}).$$

Entonces:

$$\int_0^\infty \frac{dx}{(x+1)\sqrt{x}} = \lim_{T\to\infty}\int_0^T \frac{1}{(x+1)\sqrt{x}}\,dx = \lim_{T\to\infty} 2\left(\operatorname{Arctan}(\sqrt{x})\Big|_0^T\right)$$

$$= \lim_{T\to\infty} 2\left(\operatorname{Arctan}\sqrt{T} - \operatorname{Arctan} 0\right) = \lim_{T\to\infty} 2\left(\operatorname{Arctan}\sqrt{T} - 0\right)$$

$$= 2\cdot\frac{\pi}{2} \Longrightarrow \boxed{\int_0^\infty \frac{dx}{(1+x)\sqrt{x}} = \pi}$$