Monday, February 29, 2016

How to Compute the Gaussian Integral?

1. The Classical Method: Multivariable Calculus.

Here, we assume the integral

$$I = \int_{-\infty}^{\infty} e^{-x^2}\, dx$$

converges. Since x is a dummy variable, we can write

$$I = \int_{-\infty}^{\infty} e^{-y^2}\, dy.$$

Hence:

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2}\, dx \cdot \int_{-\infty}^{\infty} e^{-y^2}\, dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dx\, dy$$

Under the change of variables:

$$x = r\cos\theta \qquad 0 < r < \infty$$
$$y = r\sin\theta \qquad 0 \le \theta < 2\pi$$

and $\dfrac{\partial(x,y)}{\partial(r,\theta)} = r$, we have $x^2+y^2 = r^2$ and

$$I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\, r\, dr\, d\theta = \int_0^{\infty} r e^{-r^2}\, dr \int_0^{2\pi} d\theta$$

$$= \int_0^{\infty} \frac{d}{dr}\left(\frac{e^{-r^2}}{-2}\right) dr \cdot 2\pi = 2\pi \left(\frac{e^{-r^2}}{-2}\right)\Bigg|_0^{\infty} =$$

$$= (-\pi)\left(\lim_{r\to\infty} e^{-r^2} - e^0\right) = \pi \cdot 1 \Rightarrow I^2 = \pi$$

$$\Rightarrow \boxed{\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}} = d =$$

... proofs were developed by Prof. J. Cruz Sampedro and Prof. M Tetlalmatzi Montiel.

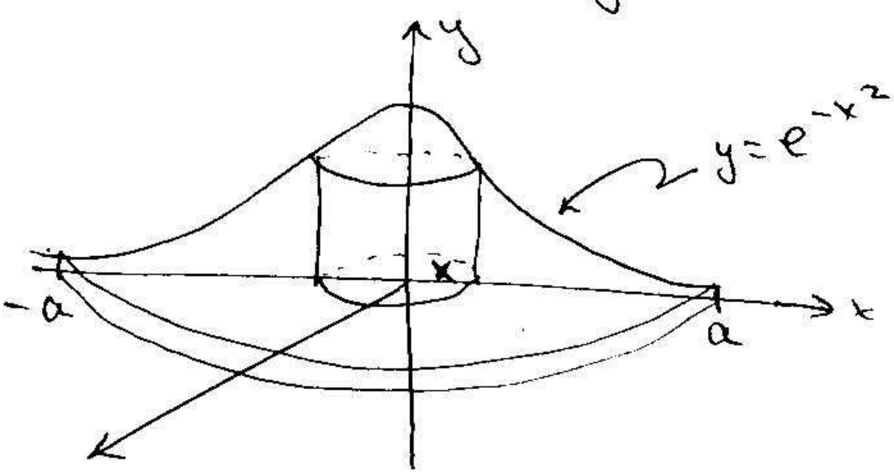
2. The method of a revolution solid

With this method, ~~convergence is proved and~~ the value of the integral is computed and the convergence follows.

Let us call $G(a)$ the revolution solid described by the rotation of $g(x)=e^{-x^2}$ about the y-axis, up to $x=a$:

$$R_a = R(g(a)) = \{(x,y) / \ x\in[0,a],\ 0\leq y\leq e^{-x^2}\}.$$

We call $g(x)=e^{-x^2}$ the gaussian function. This region is rotated about the y-axis.



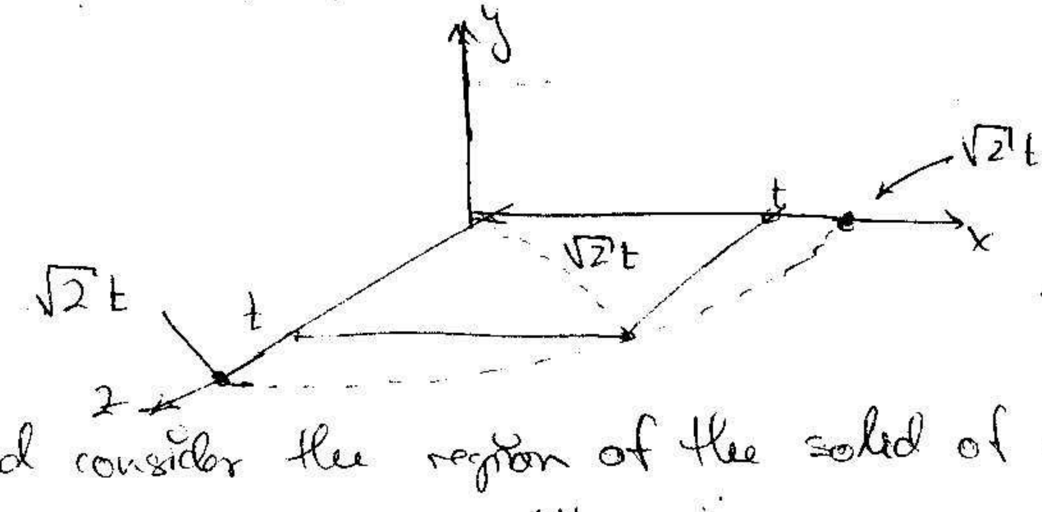
According to methods of Integral calculus, the volume of $G(a)$ is given by: (Method of shells).

$$\mathrm{Vol}\ G(a) = \int_0^a 2\pi x\, g(x)\,dx = 2\pi\int_0^a x e^{-x^2}dx = 2\pi\int_0^a \frac{d}{dx}\left(\frac{e^{-x^2}}{-2}\right)dx$$

$$= 2\pi\left(\frac{e^{-a^2}-e^0}{-2}\right) = \pi\left(1-e^{-a^2}\right)$$

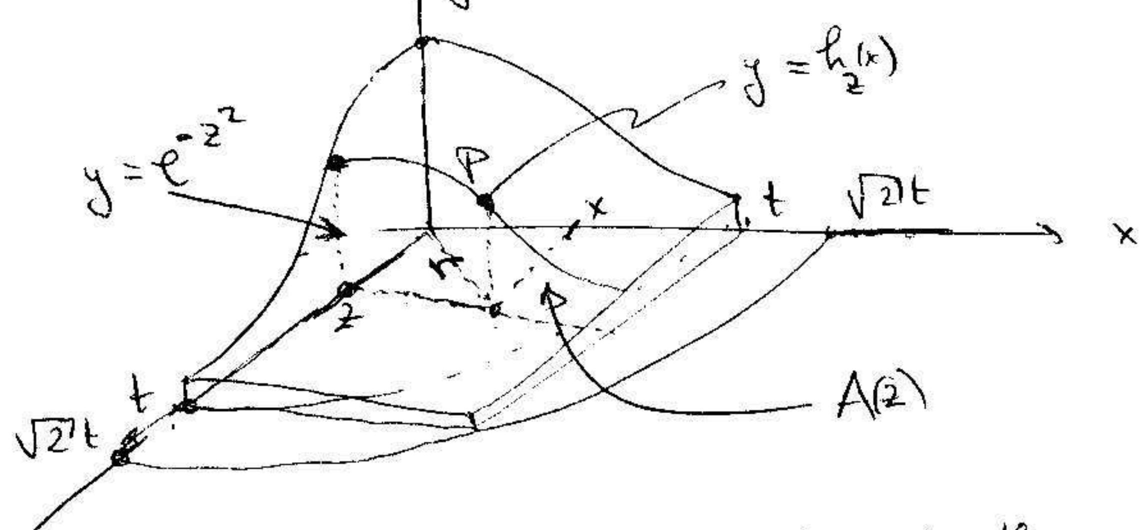
On the other hand, consider the square, in the x-z-plane

$$S_t = \{ (x,z) / 0 \le x \le t, \ 0 \le z \le t \}$$



$t^2+t^2=(\sqrt{2}t)^2$
Pythagorean Theorem

and consider the region of the solid of revolution $G(\sqrt{2}t)$,



constrained on the square S_t. Call $\mathcal{O}(t)$ this region.
We will compute its volume using the cross-section method.
Take z fixed, and consider the cross-section ~~parallel~~ parallel to the plane xy. The volume of $\mathcal{O}(t)$ is given by

$$Vol(\mathcal{O}(t)) = \int_0^t A(z)\,dz.$$

but we have to compute the area of that cross-section $A(z)$. To do this, we require to know $h_z(x)$.

The height of the Point P determines the function $h_z(x)$. Notice that, by notation

$$h_z(x) = e^{-r^2}$$

Since $r^2 = x^2 + z^2$, hence.

$$h_z(x) = e^{-(x^2+z^2)},$$

and so,

$$A(z) = \int_0^t h_z(x)\,dx = e^{-z^2}\int_0^t e^{-x^2}\,dx.$$

Call $F(t) \equiv \int_0^t e^{-x^2}\,dt$;

Hence $A(z) = e^{-z^2} F(t)$

and

$$\mathrm{Vol}(Q(t)) = \int_0^t A(z)\,dz = \int_0^t e^{-z^2} F(t)\,dz$$

$$= F(t)\int_0^t e^{-z^2}\,dz = F(t)\cdot F(t)$$

ie

$$\boxed{\mathrm{Vol}(Q(t)) = F^2(t)}$$

Now, observe that:

$$\frac{1}{4}\mathrm{Vol}(G\,A) \le \mathrm{Vol}(Q(t)) \le \frac{1}{4}\mathrm{Vol}(G\sqrt{2}\,t)$$

$$\Rightarrow \quad \frac{1}{4}\pi\left(1-e^{-t^2}\right) \le F^2(t) \le \frac{1}{4}\pi\left(1-e^{-2t^2}\right)$$

If we take $\sqrt{2}$:

$$\frac{\sqrt{\pi}}{2}\sqrt{1-e^{-t^2}} \leq F(t) \leq \frac{\sqrt{\pi}}{2}\sqrt{1-e^{-2t^2}}$$

and let $t \to \infty$, we get.

$$\frac{\sqrt{\pi}}{2} \leq \lim_{t\to\infty} F(t) \leq \frac{\sqrt{\pi}}{2}.$$

Hence, the limit, $\lim_{t\to\infty} F(t)$ exists (this proves convergence) and its value is $\sqrt{\pi}/2$ i.e.

$$\lim_{t\to\infty} F(t) = \frac{\sqrt{\pi}}{2}.$$

But; $F(t) = \int_0^t e^{-x^2}\,dx$. Hence.

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2},$$

i.e.

$$\boxed{\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}}$$

3. The method of volumes (The second method)

This method assumes the integral converges. Consider the region

$$R(g(\infty)) = R_\infty = \{(x,y) / \ 0 \leq y \leq g(x), \ x \in [0,\infty)\}$$

The volume of revolution of rotating R_∞ about the y-axis i

$$\mathrm{Vol}(G_d(\infty)) = \int^{\infty} 2\pi x\, e^{-x^2}\,dx = 2\pi \int_0^{\infty} \frac{d}{dx}\left(\frac{e^{-x^2}}{-2}\right) dx$$

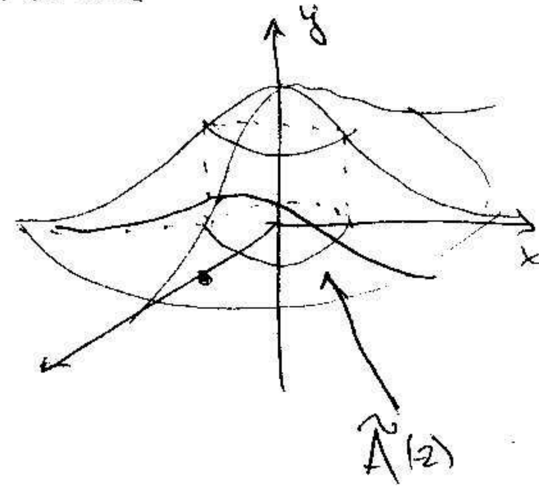
$$= 2\pi\left(\lim_{x\to\infty} \frac{e^{-x^2}}{-2} - \frac{e^{-x^2}}{-2}\Big|_{x=0}\right)$$

$$= 2\pi\left(0 + \frac{1}{2}\right) = \pi.$$

Again, let $\tilde{I} = \int_{-\infty}^{\infty} e^{-x^2}\,dx$ (here, we assume it converges)

Now, using the method of cross-section

$$\mathrm{Vol}(G(\infty)) = \int_{-\infty}^{\infty} \tilde{A}(z)\,dz$$



But $$\tilde{A}(z) = \int_{-\infty}^{\infty} h_z(x)\,dx$$

$$= \int_{-\infty}^{\infty} e^{-z^2} e^{-x^2}\,dx$$

$$= e^{-z^2} \int_{-\infty}^{\infty} e^{-x^2}\,dx = e^{-z^2}\tilde{I}.$$

Hence: $$\mathrm{Vol}(G(\infty)) = \int_{-\infty}^{\infty} \tilde{A}(z)\,dz = \tilde{I}\int_{-\infty}^{\infty} e^{-z^2}\,dz = \tilde{I}^2$$

But $\mathrm{Vol}(G(\infty)) = \pi \Rightarrow \tilde{I}^2 = \pi$

$$\Rightarrow \tilde{I} = \pi \Rightarrow \boxed{\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}}$$

4. <u>The non-obvious method</u> (Prof. Sampedro's Third Method)

We define $F(t) = \int_0^t e^{-x^2}\,dt$. The question is to ask about $\frac{d}{dt}(F^2(t))$, the rate of change of $(F^2(t))$,

$$\frac{d}{dt}(F^2(t)) = 2F(t)\frac{dF}{dt} = 2\int_0^t e^{-x^2}d\quad e^{-t^2}$$

$$= \int_0^t 2\, e^{-x^2} e^{-t^2}\,dx \underset{\uparrow}{=} \int_0^1 2e^{-s^2t^2} e^{-t^2} t\,ds$$

$x = ts$ (t - fixed, s - new variable)

$$= \int_0^1 2te^{-(s^2+1)t^2}\,ds = \int_0^1 \frac{\partial}{\partial t}\left(\frac{e^{-(1+s^2)t^2}}{-(1+s^2)}\right)ds$$

$$= \frac{d}{dt}\int_0^1 \frac{e^{-(1+s^2)t^2}}{-(1+s^2)}\,ds.$$

Integrating:

$$F^2(t) = \int_0^1 \frac{e^{-(1+s^2)t^2}}{-(1+s^2)}\,ds + C,$$

where C is the constant of Integration. To evaluate C, evaluate at $t=0$ the previous equation. Since $F(0)=0$, then:

$$0 = -\int_0^1 \frac{1}{1+s^2}\,ds + C$$

$$\Rightarrow C = \int_0^1 \frac{1}{1+s^2}\,ds = \text{Arctan}(1) - \text{Arctan}(0) = \frac{\pi}{4} - 0$$

Hence. $$F^2(t) = \frac{\pi}{4} - \int_0^1 \frac{e^{-(1+s^2)t^2}}{1+s^2}\, ds.$$

Notice that
$$\lim_{t\to\infty} \left| \frac{e^{-(1+s^2)t^2}}{1+s^2} \right| \leq \lim_{t\to\infty} \frac{e^{-t^2}}{1+s^2} = 0,$$

for any value of s, then, the convergence is uniform and we can change the limit with the integral,

and so:
$$\lim_{t\to\infty}(F^2(t)) = \frac{\pi}{4} - \int_0^1 \lim_{t\to\infty} \frac{e^{-(1+s^2)t^2}}{1+s^2}\, ds$$

$$\left(\int_0^\infty e^{-x^2} dx\right)^2 = \frac{\pi}{4}. \Rightarrow \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

$$\Rightarrow \boxed{\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}}$$