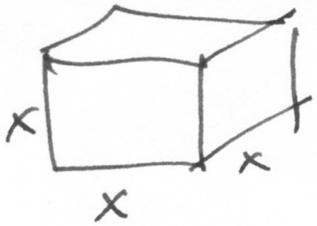
Exam #2. Differential Calculus November 13, 2014

UAM-Azcapotzalco. Fall 2014

(1.) The side of the cube is x; then



$V(x) = x^3$ is the volume.

$A(x) = 6x^2$ is the surface area

We know that $\dot{A} = \dfrac{dA}{dt} = 30\ cm^2/sec.$

Hence:

$$\dot{V} = \frac{dV}{dt} = 3x^2 \frac{dx}{dt}$$

$$\dot{A} = \frac{dA}{dt} = 12x \frac{dx}{dt}$$

Hence

$$\frac{\dot{V}}{\dot{A}} = \frac{3x^2}{12x} = \frac{x}{4} \Rightarrow \dot{V} = \frac{x}{4}\dot{A}$$

$$\Rightarrow \dot{V} = \frac{x}{4}(30) \Rightarrow \dot{V} = \frac{15}{2}x$$

If $x = 2\,cm \Rightarrow \boxed{\dfrac{dV}{dt} = 15\ cm^3/sec}$

(2.) The Mean Value Theorem

f is C^1 in $[a,b]$. Then, $\exists\, c \in [a,b]$ such that:

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Here $f(x) = x - \frac{1}{3}x^3$.

Hence: $[a,b] = [-\sqrt{3}, 0]$

$$f(b) = 0$$

$$\begin{aligned} f(a) &= (-\sqrt{3}) - \frac{1}{3}(-\sqrt{3})^3 \\ &= -\sqrt{3} + \frac{1}{3}\sqrt{3}^3 = \\ &= -\sqrt{3} + \frac{\sqrt{3}\,\sqrt{3}^2}{3} = \\ &= -\sqrt{3} + \sqrt{3} \\ &= 0. \end{aligned}$$

Hence, $\exists\, c \in [-\sqrt{3}, 0]$, such that:

$$f'(c) = \frac{0-0}{0-(-\sqrt{3})} = 0.$$

Now, $f'(x) = 1 - x^2 \Rightarrow 1 - c^2 = 0$

$\Rightarrow c = \pm 1$ since $c \in [-\sqrt{3}, 0] \Rightarrow \boxed{c = -1}$

(3) $g(x) = x^{2/3}(x+5)$. Usar el criterio de la primera derivada.

$\text{Dom}(g) = [-6,1]$

$$g'(x) = \frac{2}{3}x^{-1/3}(x+5) + x^{2/3}\cdot 1 = \frac{2}{3}\frac{x+5}{x^{1/3}} + x^{2/3}$$

$$= \frac{2(x+5)+3x}{3x^{1/3}} = \frac{5x+10}{3x^{1/3}} =$$

i.e. $\boxed{g'(x) = \frac{5(x+2)}{3x^{1/3}}}$

Puntos críticos: $g'(x) = 0$ en $x=-2$.

$g'(x)$ no existe en $x=0$.

y los puntos frontera: $x=-6$ y $x=1$

Intervalos: $(-6,-2)$, $(-2,0)$, $(0,1)$.

$g'(-3) = \frac{5(-3+2)}{3(-3)^{1/3}} > 0 \Rightarrow g(x) \nearrow$ en $(-6,-2)$

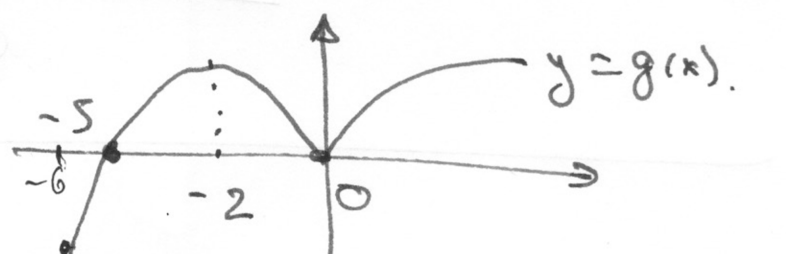
$g'(-1) = \frac{5(-1+2)}{3(-1)^{1/3}} < 0 \Rightarrow g(x) \searrow (-2,0)$

$g'(1) = \frac{5(1+2)}{3(1)^{1/3}} > 0 \Rightarrow g(x) \nearrow (0,1)$

Además:

$g(-5) = 0$

$g(0) = 0$ $\Rightarrow$



Hay máximo local: en $x = -2$

$$g(-2) = (-2)^{2/3}(-2+5)$$

$$= \sqrt[3]{4}\,(3) \simeq 4.762$$

y mínimo local en $x = 0$

$$g(0) = 0.$$

Además

Hay máximo local en $x = 1$.

$$g(1) = 6.$$

y mínimo local en $x = -6$.

$$g(-6) = -6^{2/3} \cong -3.3019.$$

Entonces:

Máximo global: g

$$g(1) = 6$$

Mínimo global:

$$g(-6) = -3.3019$$

$\boxed{4.}$ $h(x) = \frac{9}{14} x^{1/3}(x^2 - 7)$, dom$(h) = [-3, 2]$

Compute the first derivative: $h(x) = \frac{9}{14}\left(x^{7/3} - 7x^{1/3}\right)$

$\Rightarrow h'(x) = \frac{9}{14}\left(\frac{7}{3}x^{4/3} - \frac{7}{3}x^{-2/3}\right) = \frac{9}{14}\cdot\frac{7}{3}\left(x^{4/3} - \frac{1}{x^{2/3}}\right)$

$$\Rightarrow \boxed{h'(x) = \frac{3}{2}\frac{(x^2-1)}{x^{2/3}}}$$

Critical points: $h'(x) = 0$, if $\boxed{x = 1, \quad x = -1}$

$h'(x)$ does not exist, if $\boxed{x = 0}$

Boundary points: $\boxed{x = -3, \quad x = 2}$

Now, compute 2nd derivative: $h'(x) = \frac{3}{2}\left(x^{4/3} - \frac{1}{x^{2/3}}\right)$

$\Rightarrow h''(x) = \frac{3}{2}\left(\frac{4}{3}x^{1/3} + \frac{2}{3}\frac{1}{x^{5/3}}\right) = \frac{3}{2}\cdot\frac{2}{3}\left(2x^{1/3} + \frac{1}{x^{5/3}}\right)$

$= 1\cdot\left(\frac{2x^2+1}{x^{5/3}}\right)$

$$\Rightarrow \boxed{h''(x) = \frac{2x^2+1}{x^{5/3}}}$$

The numerator $2x^2 + 1 > 0$, always positive

The denominator: $x^{5/3} > 0$, when $x > 0$

$x^{5/3} < 0$, when $x < 0$

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The denominator is $\begin{cases} \text{positive, if } x > 0 \\ \text{negative, if } x < 0. \end{cases}$

$$\begin{cases} x^{5/3} > 0, \text{ if } x > 0 \\ x^{5/3} < 0, \text{ if } x < 0. \end{cases}$$

Hence $h''(x) = \dfrac{2x^2+1}{x^{5/3}} > 0$, if $x > 0$

cóncava hacia arriba

$h''(x) = \dfrac{2x^2+1}{x^{5/3}} < 0$, if $x < 0$.

cóncava hacia abajo

Hence at $x = -1$, $h'(-1) = 0$, $h''(-1) < 0$

then $h(-1) = \frac{9}{14}(-1)(1-7)$

i.e. $h(-1) = \frac{27}{7} \simeq 3.85\ldots$

is a local máximum.

And at $x = +1$, $h'(+1) = 0$, $h''(+1) > 0$,

then $h(+1) = \frac{9}{4} 1^{1/3}(1-7) =$

i.e. $h(1) = -\frac{27}{7} \cong -3.85\ldots$

is a local minimum.

Now: $h(0) = 0$

$$h(2) = \frac{9}{14} 2^{1/3}(4-7) = -2^{1/3}\frac{27}{14} \cong -2.43\ldots$$

$$h(-3) = \frac{9}{14}(-3)^{1/3}(9-7) = -\frac{9}{7}3^{1/3} \cong -1.85$$

Here:

$h(-3) \cong -1.85\ldots$

$h(-1) \cong 3.85\ldots$

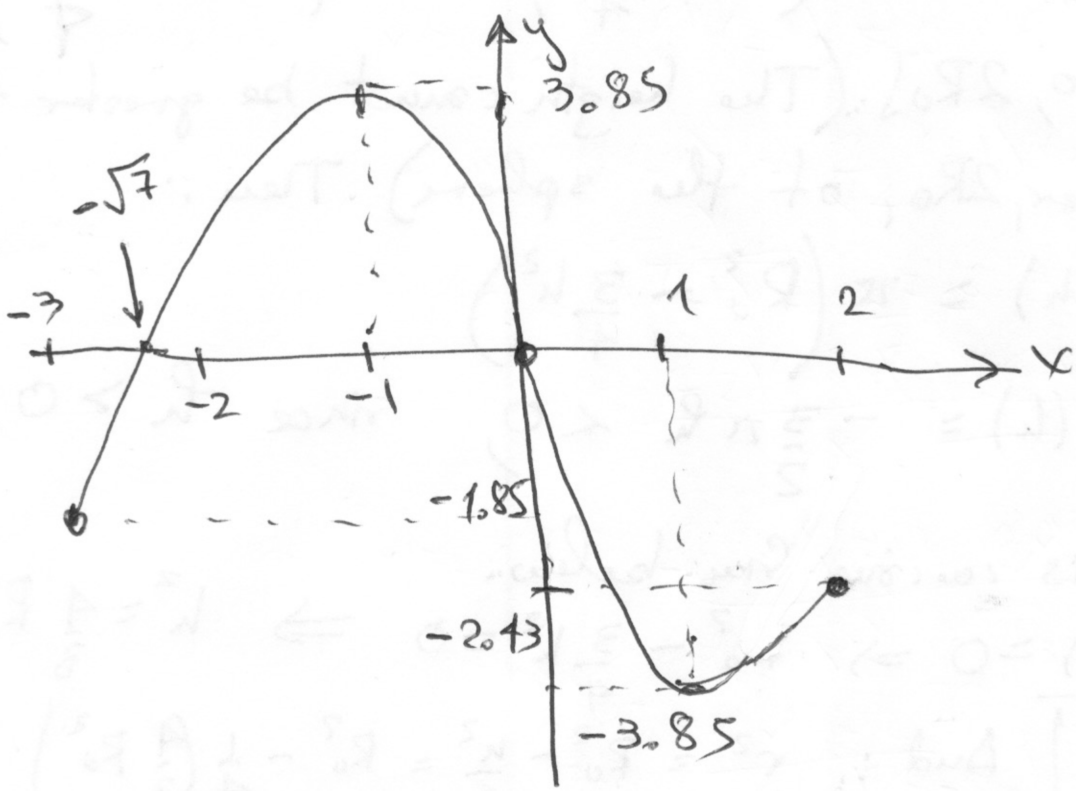
$h(0) \simeq 0$

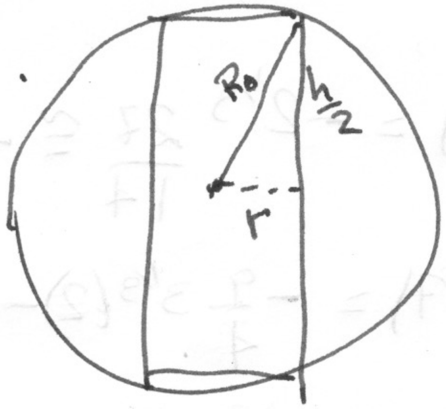
$h(1) \cong -3.85\ldots$

$h(2) \cong -2.43$

Global maximum $h(-1) \simeq 3.85\ldots$

Global minimum: $h(1) \cong -3.85$



⑤ 

r = radius of the cylinder

h = height of the cylinder

$$\boxed{r^2 + \left(\frac{h}{2}\right)^2 = R_0^2} \quad \text{- - - - - - -} (*)$$

by the Pythagorean theorem, since the ~~sphere~~ cylinder is inscribed to the sphere.

The volume of the cylinder.

$$V = \pi r^2 h.$$

From $(*)$, we have. $r^2 = R_0^2 - \frac{h^2}{4}$. Then.

$$V(h) = \pi\left(R_0^2 - \frac{h^2}{4}\right)h = \pi\left(R_0^2 h - \frac{h^3}{4}\right)$$

with $h \in [0, 2R_0]$. (The heigth cannot be greater than the diameter, $2R_0$, of the sphere). Then.

$$V'(h) = \pi\left(R_0^2 - \frac{3}{4}h^2\right)$$

$$V''(h) = -\frac{3}{2}\pi h < 0, \quad \text{since } h > 0.$$

Then, $V(h)$ is concave from below.

Now $V'(h) = 0 \Rightarrow R_0^2 - \frac{3}{4}h^2 = 0 \Rightarrow h^2 = \frac{4}{3}R_0^2$

$\Rightarrow \boxed{h = \frac{2R_0}{\sqrt{3}}}$ And: $r^2 = R_0^2 - \frac{h^2}{4} = R_0^2 - \frac{1}{4}\left(\frac{4}{3}R_0^2\right)$

$r^2 = \frac{2}{3}R_0^2 \Rightarrow \boxed{r = \sqrt{\frac{2}{3}}\,R_0}$ are the dimensions of the cylinder

Its volume $V = \pi r^2 h = \pi\left(\frac{2}{3}R_0^2\right)\frac{2}{\sqrt{3}}R_0 = \boxed{\frac{4\pi R_0^3}{3\sqrt{3}}}$