Intro a las Ecuaciones Diferenciales Ordinarias Wedn, July 8, 2015

Examen #2 UAM-Azcapotzalco.

---

① This is a: 1) linear, 2) const. coefficients, 3) homogeneous, ODE

$\Rightarrow y(t) = e^{rt} \Rightarrow r^2 + 4r + 13 = 0 \Rightarrow r_{1,2} = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 13}}{2}$

$\Rightarrow r_{1,2} = \frac{-4 \pm 2\sqrt{4-13}}{2} = \frac{-4 \pm 2\sqrt{-9}}{2} = \frac{-4 \pm 2(i3)}{2}$

$\boxed{r_{1,2} = -2 \pm 3i}$

$\Rightarrow \quad y_1(t) = e^{-2t} e^{3it}, \quad y_2(t) = e^{-2t} e^{-3it}$

$y(t) = C_1 e^{2t} e^{3it} + C_2 e^{-2t} e^{-3it}$

$= e^{-2t}(C_1 e^{3it} + C_2 e^{-3it})$

Using Euler's formula, we got:

$$\boxed{y(t) = e^{-t}(A\cos(3t) + B\sin(3t))}$$

(2) This is a 1) linear, 2) const coefficients, 3) homogeneous ODE

$\Rightarrow y(t) = e^{rt} \Rightarrow r^2 + 4r + 4 = 0 \Rightarrow r = \frac{-4 \pm \sqrt{4^2 - 4\cdot 4}}{2}$

$\Rightarrow r_{1,2} = -2 \Rightarrow r_1 = r_2 = -2$

$\Rightarrow y_1(t) = e^{-2t};\ y_2(t) = t e^{-2t}$

$$\boxed{y(t) = C_1 e^{-2t} + C_2 t e^{-2t}}$$

Or $\boxed{y(t) = (C_1 + C_2 t) e^{-2t}}$

Now: $y'(t) = C_2 e^{-t} + (C_1 + C_2 t) e^{-2t}(-2)$

Now $5 = y(0) = C_1 \Rightarrow \boxed{C_1 = 5}$

$3 = y'(0) = C_2 + (-2)C_1 \quad 3 = C_2 + 10$

$\Rightarrow C_2 = 13 \Rightarrow \boxed{y(t) = (5 + 13t) e^{-2t}}$

¿$y(t) = 0$? Since $e^{-2t} \neq 0 \Rightarrow y(t) = (5 + 13t) e^{-2t} = 0$

$\Rightarrow 5 + 13t = 0$

$\Rightarrow \boxed{t = -\frac{5}{13}}$

Conditions $y(0) = 4$
$y'(0) = -10$:

$C_1 = 4$, and $C_2 = -2$

$y'(0) = C_2 + (-2)C_1 = -2 - 2\cdot 4 = -10$

$y(t) = 0 \Rightarrow t = -\frac{C_1}{C_2} = \frac{-4}{-2} = 2$

$-6 = C_2 + (-2)C_1$

$2C_1 - 6 = C_2$

$2\cdot 4 - 6 = C_2$

$2 = C_2$

$\mathbb{Z} =$

(3). Consider $y_2(t) = v(t)\,y_1(t) \Rightarrow y_2(t) = e^t v(t)$

$$\Rightarrow y_2' = e^t(v+v') \quad \text{and} \quad y_2'' = e^t(v+2v'+v'')$$

$$(t-1)e^t(v+2v'+v'') - te^t(v+v') + e^t v = 0$$

Since $e^t \neq 0$, divide by e^t:

$$(t-1)(v+2v'+v'') - t(v+v') + v = 0$$

Grouping terms in v'', v' and v:

$$v''(1-t) + v'(2(t-1)-t) + v\Big(\underbrace{(t-1)-t+1}_{=0}\Big) = 0$$

$\Rightarrow (t-1)v'' + (t-2)v' = 0$ Define: $V = \frac{dv}{dt}$ (Reduction of order)

$\Rightarrow (t-1)\frac{dV}{dt} + (t-2)V = 0$. Separate variables: $\frac{1}{V}\frac{dV}{dt} = -\left(\frac{t-2}{t-1}\right)$

$$\Rightarrow \frac{1}{V}\frac{dV}{dt} = -\left(\frac{t-1-1}{t-1}\right) = -\left(1-\frac{1}{t-1}\right) = -1+\frac{1}{t-1}.$$

$$\Rightarrow \frac{d}{dt}(\operatorname{Log} V) = \frac{d}{dt}\left(-t+\operatorname{Log}|t-1|+C_1\right) \Rightarrow \operatorname{Log} V = -t+\operatorname{Log}|t-1|+C_1$$

$$\Rightarrow V = e^{-t}\cdot(t-1)\cdot \underbrace{e^{C_1}}_{=C_2} \Rightarrow \frac{dv}{dt} = C_2\left(te^{-t}-e^{-t}\right)$$

$$\Rightarrow v(t) = C_2\int te^{-t}\,dt - C_2\int e^{-t}\,dt = C_2\left[-te^{-t}+\int e^{-t}\,dt\right] + C_2e^{-t} + C_3$$

$$= -C_2(t+1)e^{-t} + C_2e^{-t} + C_3 = -C_2te^{-t}\underbrace{-C_2e^{-t}+C_2e^{-t}}_{=0}+C_3$$

$$= -C_2te^{-t} + C_3. \Rightarrow y_2(t) = e^t\left(-C_2te^{-t}+C_3\right)$$

$\Rightarrow y_2(t) = -C_2t + \underbrace{C_3e^t}_{\text{Repeats } y_1(t) \Rightarrow C_3=0}$ Hence: $y_2(t) = -C_2t \Rightarrow y_2(t) = t$ ($C_2=-1$)

$$\Rightarrow \boxed{y(t) = C_1e^t + C_2t}$$

(4) Solve: $y'' - 5y' + 6y = 0. \Rightarrow r^2 - 5r + 6 = 0 \Rightarrow (r-2)(r-3)$

$y_1(t) = e^{2t}$, $y_2(t) = e^{3t}$

$y_p(t) = (At + B)e^t$. It does not repeat solutions to homogeneous equation.

$$\Rightarrow y_p'(t) = Ae^t + (At+B)e^t = (At + A + B)e^t$$

$$y_p''(t) = 2Ae^t + (At+B)e^t = (At + 2A + B)e^t$$

$$\Rightarrow (At + 2A + B)e^t - 5(At + (A+B))e^t + 6(At+B)e^t = te^t$$

Divide by e^t:

$$(At + 2A + B) - 5(At + A + B) + 6(At + B) = t.$$

Grouping linear terms (in t) and constant terms:

$$(A - 5A + 6A)t + (2A + B - 5(A+B) + 6B) = t$$

$$(2A)t + (-3A + 2B) = t. \Rightarrow \begin{cases} 2A = 1 \Rightarrow \\ -3A + 2B = 0 \end{cases}$$

$$\Rightarrow \boxed{A = \frac{1}{2}} \text{ and } -\frac{3}{2} + 2B = 0 \Rightarrow 2B = 0 + \frac{3}{2} = \frac{3}{2}$$

$$\Rightarrow \boxed{B = \frac{3}{4}}$$

$$\Rightarrow \boxed{y(t) = C_1e^{2t} + C_2e^{3t} + \left(\frac{t}{2} + \frac{3}{4}\right)e^t}$$

(5) Solve $y'' - 5y' + 6y = 0 \Rightarrow y = e^{rt} \Rightarrow r^2 - 5r + 6 = 0$

$\Rightarrow (r-2)(r-3) = 0 \Rightarrow y(t) = C_1 e^{2t} + C_2 e^{3t}.$

$= C_1 y_1(t) + C_2 y_2(t)$

Particular solution.

$$y_p(t) = A(t)\, y_1(t) + B(t)\, y_2(t)$$

We require $a(t) = 1$, $g(t) = te^t$, $W = \begin{vmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{vmatrix} = 3e^{5t} - 2e^{5t} = e^{5t}$

Hence $A(t) = -\int \frac{g(t)\, y_2(t)}{a(t)\, W[y_1, y_2](t)}\, dt = -\int \frac{te^t e^{3t}}{1 \cdot e^{5t}}\, dt$

$= -\int t e^{-t}\, dt = -\left[-te^{-t} + \int e^{-t}\, dt\right] = -\left[-te^{-t} - e^{-t}\right]$

i.e.

$A(t) = (+t + 1)e^{-t}$

$B(t) = \int \frac{g(t)\, y_1(t)}{a(t)\, W[y_1, y_2](t)}\, dt = \int \frac{te^t e^{2t}}{1 \cdot e^{5t}}\, dt = \int t e^{-2t}\, dt = \frac{te^{-2t}}{-2} + \frac{1}{2}\int e^{-2t}\, dt$

$= \frac{te^{-2t}}{-2} - \frac{1}{4} e^{-2t}$

$= -\frac{1}{2}\left(t + \frac{1}{2}\right)e^{-2t}$

$\Rightarrow \quad y_p(t) = (+t + 1)e^{-t} \cdot e^{2t} + \left(\frac{-1}{2}\right)\left(t + \frac{1}{2}\right)e^{-2t} e^{3t}$

$= \left[(+t + 1) + \left(-\frac{t}{2}\right) - \frac{1}{4}\right] e^t$

$= \left(+\frac{1}{2}t + \frac{3}{4}\right) e^t$

(6) $y(t) = C_1 e^{2t} + C_2 t e^{2t} + C_3 t^2 e^{2t} + C_4 t^3 e^{2t} + C_5 t^4 e^{2t} + t^5 e^{2t}$.

The solution $y_1(t) = e^{2t}$ is multiplied by t up to t^4.

This means $r=2$ is a multiple root of order 5.

$$(r-2)^5 = 0$$

$$r^5 + 5r^4(-2) + 10r^3(-2)^2 + 10r^2(-2)^3 + 5r(-2)^4 + (-2)^5 = 0.$$

$$r^5 - 10r^4 + 4r^3 - 80r^2 + 80r - 32 = 0$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

$$\Rightarrow \frac{d^5}{dt^5}y - 10\frac{d^4}{dt^4}y + 4\frac{d^3}{dt^3}y - 80\frac{d^2}{dt^2}y + 80\frac{dy}{dt} - 32y = g(t)$$

Here $g(t)$ should be e^{2t}.

Since $Y_1(t) = Ae^{2t}$ repeats $C_1 y_1(t) = C_1 e^{2t}$.

and we have to multiply up to t^4 to repeat $y_5(t) = C_5 t^4 e^{2t}$ and, hence $y_p(t) = t^5 e^{2t}$,

does not repeat. $y_1(t), y_2(t), y_3(t), y_4(t)$ or $y_5(t)$.

$$\Rightarrow \boxed{y^{(5)} - 10y^{(4)} + 4y^{(3)} - 80y'' + 80y' - 32y = e^{2t}}$$

[Remark: If $g(t) = t^5 e^{2t} \Rightarrow Y_1(t) = (A_5 t^5 + \dots + A_0)e^{2t}$, but this is a wrong trial since $(A_4 t^4 + \dots + A_0)e^{2t}$ repeats homog. sol. $\Rightarrow y_p(t) = t^5(A_5 t^5 + \dots + A_0)e^{2t}$]

7. $y'' - 2y' + 5y = 0 \Rightarrow r^2 - 2r + 5 = 0$

$\Rightarrow (r - ?)(r - ?) = 0$? $\quad r = \dfrac{2 \pm \sqrt{2^2 - 4 \cdot 5}}{2}$

$r_{1,2} = \dfrac{2 \pm 2\sqrt{1-5}}{2} = 1 \pm \sqrt{-4} = 1 \pm 2i$

$\Rightarrow y(t) = C_1 e^t e^{2it} + C_2 e^t e^{-2it} \quad e^t(C_1 e^{2it} + C_2 e^{-2it})$

By Euler's formula

$y(t) = e^t(A\cos(2t) + B\sin(2t))$

solves the homogeneous equation.

First trial $Y_1(t) = e^t A\cos(2t) + e^t B\sin(2t)$

but repeats homogeneous equation then, multiply by t:

$$\boxed{y_p(t) = te^t\left(A\cos(2t) + B\sin(2t)\right)}$$

(8) $y'' + y = \sin t + t\cos t + e^{(\log 10)t}$.

Solution to homogeneous: $y_h'' + y_h = 0$

$$y_h(t) = A\cos t + B\sin t.$$

Using superposition principle

$$(y_p)_1(t) = A\cos t + B\sin t.$$

$$(y_p)_2(t) = (\alpha_1 t + \alpha_0)\cos t + (\beta_1 t + \beta_0)\sin t$$

$$(y_p)_3(t) = C e^{(\log 10)t} = C\,10^t$$

If we add them: $y_{p,1} + y_{p,2} + y_{p,3}$, we get

$$y_p(t) = (\alpha_1 t + \underbrace{\alpha_0 + A}_{\tilde{\alpha}_0})\cos t + (\beta_1 t + \beta_0 + B)\,\cancel{\cos}\,\sin t + C\,10^t$$

$$= (\alpha_1 t + \tilde{\alpha}_0)\cos t + (\beta_1 t + \tilde{\beta}_0)\sin t + C\,10^t.$$

But $\tilde{\alpha}_0 \cos t + \tilde{\beta}_0 \sin t$, repeats the solution to the homogeneous equation. Then, multiply by t:

$$\boxed{y_p(t) = t\left[(\alpha_1 t + \tilde{\alpha}_0)\cos t + (\beta_1 t + \tilde{\beta}_0)\sin t\right] + C\,10^t}$$