Examen Global Ecuaciones Diferenciales Ordinarias.

Vespertino. 2 Viernes 24 de julio de 2015

I. Primera Parte.

(1.) We have the initial value problem:

$$y' - y = e^{2t} y^3$$

$$y(0) = 1.$$

This is a Bernoulli equ. If $v^\beta = y$, differentiating we then have: $\beta v^{\beta-1} \frac{dv}{dt} = \frac{dy}{dt} = y + e^{2t} y^3 = v^\beta + e^{2t} v^{3\beta}$, ie:

i.e.

$$\beta \frac{dv}{dt} = \frac{v^\beta}{v^{\beta-1}} + e^{2t} \frac{v^{3\beta}}{v^{\beta-1}} = v + e^{2t} v^{2\beta+1}$$

Take $2\beta+1 = 0 \Rightarrow \frac{dv}{dt} - \frac{1}{\beta} v = \frac{1}{\beta} e^{2t}$.

and $v^\beta_{(0)} = y(0) = 1 \Rightarrow \therefore v(0) = 1.$

This is a first order linear I.V.P.

Since $2\beta + 1 = 0 \Rightarrow \beta = -\frac{1}{2} \Rightarrow \left\{ \begin{array}{l} \frac{dv}{dt} + 2v = -2e^{2t} \\ v(0) = 1. \end{array} \right.$

Hence, the integrating factor is $\mu(t) = e^{\int 2dt} = e^{2t}$

$$\Rightarrow \mu \frac{dv}{dt} + 2\mu v = -2\mu e^{2t}.$$

$$\frac{d}{dt}(\mu v) = -2\mu(t) e^{2t} dt.$$

$$\mu(t) v(t) - \mu(0) v(0) = -2 \int_0^t \mu(s) e^{2s} ds$$

$$\Rightarrow v(t) = -\frac{2}{\mu(t)} \int_0^t \mu(s) e^{2s} ds + \frac{\mu(0) v(0)}{\mu(t)}.$$

i.e. $v(t) = -\frac{2}{e^{2t}} \int_0^t e^{2s} e^{2s} ds + \frac{1 \cdot 1}{e^{2t}} = -\frac{2}{e^{2t}} \int_0^t e^{4s} ds + \frac{1}{e^{2t}}$

$$v(t) = -\frac{2}{e^{2t}} \frac{e^{4s}}{4}\Big|_0^t + e^{-2t} = -\frac{1}{e^{2t}} \frac{e^{4t}-1}{2} + e^{-2t}.$$

$$= -\frac{e^{2t} - e^{-2t}}{4} + e^{-2t} = -\frac{e^{2t}}{2} + \frac{e^{-2t}}{2} + e^{-2t}$$

$$\Rightarrow \quad \boxed{v(t) = -\frac{e^{2t}}{2} + \frac{3}{2}e^{-2t}}$$

Now $y = v^{\beta} = v^{-1/2} \Rightarrow \boxed{y(t) = \dfrac{1}{\sqrt{-\frac{e^{2t}}{2} + \frac{3}{2}e}}}$

Alternatively, $\dfrac{dv}{dt} + 2v = -2e^{2t}$.

The integrating factor $\mu(t) = e^{\int 2\,dt} = e^{2t}$.

$$\Rightarrow v(t) = \frac{1}{\mu(t)} \int \mu(t) g(t) + \frac{C}{\mu(t)} = \frac{1}{e^{2t}} \int e^{2t}(-2e^{2t})\,dt + \frac{C}{e^{2t}}$$

$$= \frac{-2}{e^{2t}} \int e^{4t}\,dt + Ce^{-2t} = -\frac{2}{e^{2t}} \frac{e^{4t}}{4} + Ce^{-2t} \Rightarrow v(t) = -\frac{e^{2t}}{2} + Ce^{-2t}$$

$$\boxed{v(t) = -\frac{e^{2t}}{2} + Ce^{-2t}}$$

$$\Rightarrow \quad y(t) = v^{-1/2}(t) = \left(\sqrt{-\frac{e^{2t}}{2} + Ce^{-2t}}\right)^{-1}$$

$$1 = y(0) = \left(\sqrt{-\frac{1}{2} + C}\right)^{-1} \Rightarrow 1 = -\frac{1}{2} + C \Rightarrow \boxed{C = \frac{3}{2}}$$

$$\Rightarrow \quad \boxed{y(t) = \frac{1}{\sqrt{-\frac{e^{2t}}{2} + \frac{3}{2}e^{-2t}}}}$$

② This equation is ~~not~~ exact:

$$(x+3\cos y) - (x\sin y)\frac{dy}{dx} = 0$$

$M(x,y) = x + 3\cos y$

$N(x,y) = -x\sin y$ $\qquad \frac{\partial M}{\partial y} = -3\sin y \ ; \ \frac{\partial N}{\partial x} = -\sin y$

$M_y \neq N_x \Rightarrow$ it is <u>not</u> exact.

Now $\frac{M_y - N_x}{N} = \frac{(-3\sin y)-(-\sin y)}{-x\sin y} = \frac{-2\sin y}{-x\sin y} = \frac{2}{x}$

Then, $\mu = \mu(x)$ is a function of x only:

$$\frac{1}{\mu}\frac{d\mu}{dx} = \frac{2}{x} \Rightarrow \frac{d}{dx}\log|\mu| = 2\frac{d}{dx}\log|x|$$
$$= \frac{d}{dx}\log(x^2)$$

$\Rightarrow \log|\mu| = \log|x^2| + C$. Take $C=0 \Rightarrow \boxed{\mu(x) = x^2}$

$\Rightarrow \mu(x+3\cos y) - \mu(x\sin y)\frac{dy}{dx} = 0$ is <u>now</u> <u>exact</u>.

I.e.

$$(x^3 + 3x^2\cos y) - (x^3\sin y)\frac{dy}{dx} = 0 \ldots\ldots (\star)$$

Hence: $\frac{\partial \Psi}{\partial x} = x^3 + 3x^2\cos y \ldots\ldots (*)$

$\frac{\partial \Psi}{\partial y} = -x^3\sin y \ldots\ldots (**)$

$\frac{\partial \Psi}{\partial y} = -x^3\sin y \Rightarrow \Psi(x,y) = x^3\cos y + f(x) \ldots\ldots (***)$

Tomando la derivada en $(***)$: $\frac{\partial}{\partial x}\Psi = 3x^2\cos y + f'(x)$. y

comparando con $(*)$:

$$f'(x) = x^3 \Rightarrow f(x) = \frac{x^4}{4} + \text{const}$$

$$\Rightarrow \boxed{\Psi(x,y) = x^3\cos y + \frac{x^4}{4} + \text{const}} \ldots\ldots (\star\star)$$

Now, substitute (*) and (**) in (☆)

$$\frac{\partial \Psi}{\partial x} + \frac{\partial \Psi}{\partial y}\frac{dy}{dx} = 0$$

i.e. $$\frac{d}{dx}\left(\Psi(x, y(x))\right) = 0$$

i.e. $$\Psi(x, y) = \mathcal{C}$$, where $\mathcal{C}$ = constante

Using (**), then:

$$x^3 \cos y + \frac{x^4}{4} + \underbrace{\text{const}}_{} = \mathcal{C}$$

$$\Rightarrow \boxed{x^3 \cos y + \frac{x^4}{4} = C}$$ is the implicit solution

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Alternatively $$\frac{\partial \Psi}{\partial x} = x^3 + 3x^2 \cos y$$

$$\Rightarrow \Psi(x,y) = \frac{x^4}{4} + x^3 \cos y + \text{const}$$

and $$\Psi(x,y) = \boxed{\frac{x^4}{4} + x^3 \cos y = C}$$

Same answer

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(3) $\frac{dy}{dx} = \frac{8x - x^3 e^{-x^2}}{e^{-x^2}(4+2y)}$, $y(0) = 1$

Notice:

$$\frac{dy}{dx} = \frac{8x - x^3 e^{-x^2}}{e^{-x^2}(4+2y)} = \frac{e^{x^2}(8x - x^3 e^{-x^2})}{(4+2y)} = \frac{8xe^{x^2} - x^3}{4+2y}$$ and

this is a separable equation:

$$(4+2y)\frac{dy}{dx} = 8x^3 - x^3$$

$$\int (4+2y)\frac{dy}{dx}\,dx = \int 8xe^{x^2} - x^3\,dx \Rightarrow (4y + y^2) = \int 4(2xe^{x^2}) - x^3\,dx$$

$$= \qquad \Rightarrow 4y + y^2 = 4\int \frac{d}{dx}(e^{x^2}) - \frac{x^4}{4} + C$$

$$\boxed{4y + y^4 = 4e^{x^2} - \frac{x^4}{4} + 1} \quad 4 + 1 = C + 4 \quad \boxed{C = 1} \quad y(0) = 1 \Leftarrow \boxed{4y + y^2 = 4e^{x^2} - \frac{x^4}{4} + C}$$

is ~~the~~ implicit solution

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(4) We must solve $\frac{dM}{dt} = -kM$, $M(0) = 100$gr.

$$\Rightarrow M(t) = M(0)e^{-kt} \Rightarrow \boxed{M(t) = 100e^{-kt}\ \text{gr}}$$

(a) $M(100) = 75$g

$$100e^{-100k} = 75 \Rightarrow \frac{100}{75} = e^{100k} \Rightarrow \frac{4}{3} = e^{100k}$$

$$\Rightarrow \log\left(\frac{4}{3}\right) = 100k \Rightarrow \boxed{k = \frac{1}{100}\log\left(\frac{4}{3}\right)} \simeq 0.0028769\ \frac{1}{\text{año}}.$$

(c) $$M(200) = 100e^{-200k} = 100e^{-2\log\left(\frac{4}{3}\right)} = 100e^{\log\left(\frac{4}{3}\right)^{-2}} = 100\left(\frac{4}{3}\right)^{-2}$$

$$= 100\left(\frac{3}{4}\right)^2 = \frac{900}{16}\ \text{gr.} \Rightarrow \boxed{M(200) = 56.25\text{gr}}$$

(b) Vida media: $\frac{M(0)}{2} = M(0)e^{-kT} \Rightarrow e^{kT} = 2$

$\Rightarrow kT = \text{Log}\,2 \Rightarrow \boxed{T = \frac{1}{k}\text{Log}\,2}$

ie $T = \frac{100\,\text{Log}\,2}{\text{Log}(4/3)}$ años:

$\boxed{T \simeq 241 \text{ años.}}$

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(5) We have to solve:

$$\frac{dP}{dt} = rP$$
$$P(0) = 100$$

$P(t) = P(0)e^{rt} \Rightarrow P(t) = 100e^{rt}$

(a) We know: $P(100) = 10{,}000$

$\Rightarrow 100e^{100r} = 10{,}000 \Rightarrow e^{100r} = 100.$

$\Rightarrow 100r = \text{Log}\,100 \Rightarrow r = \frac{1}{100}\text{Log}(10)^2 \Rightarrow r = \frac{2}{100}\log(10)$

$\Rightarrow \boxed{r = \frac{1}{50}\text{Log}\,10 \;/\text{año}} \simeq 0.046/\text{año}.$

(b) $P(T) = 3P(0) \Rightarrow P(0)e^{rT} = 3\cdot P(0) \Rightarrow e^{rT} = 3$

$\Rightarrow rT = \text{Log}\,3 \Rightarrow T = \frac{1}{r}\text{Log}\,3 = 50\frac{\text{Log}\,3}{\text{Log}\,10}$ $\boxed{T = 50\frac{\text{Log}\,3}{\text{Log}\,10} \text{ años}}$

$\boxed{T = 23.85 \text{ años}}$

(c) $P(200) = 100e^{200r} = 100e^{4\,\text{Log}\,10} =$

$= 100e^{\text{Log}(10^4)} = 100\cdot 10^4 = 10^6$ $\boxed{P(200) = 10^6 \text{ inhabitants}}$

$\boxed{P(200) = 1 \text{ millon inhabitants}}$

SEGUNDA PARTE

(1) $4y'' + y = 2\sec\left(\frac{t}{2}\right)$. We must use variation of constants

Homogeneous equation:

$$y_h'' + \frac{1}{4}y_h = 0 \Rightarrow r^2 + \frac{1}{4} = 0 \Rightarrow r_{1,2} = \mp\frac{i}{2}$$

$$\Rightarrow \quad y_h(t) = C_1\cos\left(\frac{t}{2}\right) + C_2\sin\left(\frac{t}{2}\right).$$

$y_1(t) = \cos\left(\frac{t}{2}\right)$; $y_2(t) = \sin\left(\frac{t}{2}\right)$: $W[y_1,y_2](t) = \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$

$$\Rightarrow W[y_1,y_2](t) = \det\begin{pmatrix} \cos\frac{t}{2} & \sin\frac{t}{2} \\ -\frac{1}{2}\sin\frac{t}{2} & \frac{1}{2}\cos\frac{t}{2} \end{pmatrix} = \frac{1}{2}\cos^2\frac{t}{2} - \left(-\frac{1}{2}\sin^2\left(\frac{t}{2}\right)\right) = \frac{1}{2}\left(\cos^2\frac{t}{2} + \sin^2\frac{t}{2}\right)$$

$$\Rightarrow \boxed{W[y_1,y_2](t) = \frac{1}{2}}$$

And $a(t) = 4$:

$g(t) = 2\sec\left(\frac{t}{2}\right)$

Particular solution: $y_p(t) = A(t)\cos\left(\frac{t}{2}\right) + B(t)\sin\left(\frac{t}{2}\right)$

$$A(t) = -\int \frac{2\sec(t/2)\sin(t/2)}{4\cdot 1/2}\,dt = -\int \frac{\sin(t/2)}{\cos(t/2)}\,dt = 2\log\left|\cos\left(\frac{t}{2}\right)\right|$$

$$B(t) = \int 2\,\frac{\sec(t/2)\cos(t/2)}{4\cdot 1/2}\,dt = \int 1\,dt = t.$$

The general solution is:

$$\boxed{y(t) = C_1\cos\left(\frac{t}{2}\right) + C_2\sin\left(\frac{t}{2}\right) + 2\log\left|\cos\left(\frac{t}{2}\right)\right|\cos\left(\frac{t}{2}\right) + t\sin\left(\frac{t}{2}\right)}$$

(2) Solution to the homogeneous equation:

$$y'' - 3y' - 4y = 0$$

$$r^2 - 3r - 4 = 0$$

$$(r+1)(r-4) = 0 \Rightarrow y_1(t) = e^{-t}$$
$$y_2(t) = e^{4t}$$

Particular solution:

$$y_p(t) = Ae^t \cos(2t) + Be^t \sin(2t).$$

$$y_p'(t) = e^t(A\cos 2t + B\sin 2t) + e^t(-2A\sin(2t) + 2B\cos(2t))$$

$$= e^t((A+2B)\cos(2t) + (B-2A)\sin(2t)).$$

$$y_p''(t) = e^t((A+2B)\cos 2t + (B-2A)\sin(2t)) +$$
$$+ e^t(-2(A+2B)\sin(2t) + 2(B-2A)\cos(2t)).$$

$$= e^t((-3A+4B)\cos(2t) + (-4A-3B)\sin(2t)).$$

Hence
$$y_p'' - 3y_p' - 4y_p = -8e^t\cos 2t.$$

becomes:

$$e^t[(-3A+4B)\cos(2t) + (-4A-3B)\sin(2t)] - 3e^t[(A+2B)\cos(2t) + (B-2A)\sin(2t)] +$$
$$+ -4e^t[A\cos 2t + B\sin 2t] = -8e^t\cos 2t.$$

$$\Rightarrow e^t[(-10A-2B)\cos(2t) + (2A-10B)\sin(2t)] = -8e^t\cos 2t.$$

$$\Rightarrow \begin{cases} -10A - 2B = -8 \\ 2A - 10B = 0 \end{cases} \Rightarrow A = 5B \Rightarrow \boxed{\begin{array}{l} A = \frac{10}{13} \\ B = \frac{2}{13} \end{array}}$$

$$\boxed{y(t) = C_1 e^{-t} + C_2 e^{4t} + \left(\frac{10}{13}\cos 2t + \frac{2}{13}\sin 2t\right)e^t}$$

③ $t^2y'' + ty' - y = 0$

tiene solución $y_1(t) = t$.

$\Rightarrow$ Reducción de orden: $y_2(t) = v(t)\,y_1(t) = tv$.

$$y_2' = tv' + v$$

$$y_2'' = tv'' + 2v'$$

$$\Rightarrow \quad t^2(tv'' + 2v') + t(tv' + v) - (tv) = 0$$

$$t^3v'' + 2t^2v' + t^2v' + tv - (tv) = 0$$

$$\Rightarrow \quad t^3v'' + 3t^2v' = 0 \qquad v' = V$$

$$\Rightarrow t^3V' + 3t^2V = 0 \quad \Rightarrow \quad \frac{dV}{dt} = -\frac{3}{t}V \quad \Rightarrow \quad \frac{dV}{V} = \frac{-3\,dt}{t}$$

$$\Rightarrow \quad \log V = -3\log t \quad \Rightarrow \quad V = t^{-3} \Rightarrow \frac{dv}{dt} = t^{-3}$$

$$y_2 = v(t)\cdot t$$

$$\Rightarrow \quad v(t) = \frac{t^{-2}}{-2} \quad \Rightarrow \quad y_2(t) = \frac{t^{-2}}{-2}\cdot t = -\frac{t^{-1}}{2}$$

But $-\frac{1}{2}$ is a constant: $\Rightarrow y_2(t) = C_2 t^{-1}$

Thus: $y_1(t) = t$ and $y_2(t) = \frac{1}{t}$

$$\boxed{y(t) = C_1 t + \frac{C_2}{t}}$$ is the solution to the homogeneous solution.

(4) Homogeneous solution:-

$$y_h'' + y_h' - 2y_h = 0$$

$$r^2 + r - 2 = 0 \Rightarrow r_1 = 1, \; r_2 = -2$$

$$\Rightarrow y_1(t) = e^t, \; y_2(t) = e^{-2t}$$

General solution:

$$\boxed{y(t) = C_1 e^t + C_2 e^{-2t} + \left(t + \frac{1}{2}\right)}$$

Now:

$$y'(t) = C_1 e^t - 2C_2 e^{-2t} + 1$$

$$\Rightarrow \left.\begin{aligned} C_1 + C_2 + \frac{1}{2} &= 1 \\ C_1 - 2C_2 + 1 &= \frac{3}{4} \end{aligned}\right\} \Rightarrow 3C_1 - \frac{1}{2} = \frac{1}{4}$$

$$\Rightarrow 3C_1 = \frac{3}{4}$$

$$\boxed{C_1 = \frac{1}{4}}$$

and $C_2 = 1 - \frac{1}{2} - C_1 = 1 - \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$ $\quad \boxed{C_2 = \frac{1}{4}}$

$$\Rightarrow \boxed{y(t) = \frac{1}{4} e^t + \frac{1}{4} e^{-2t} + \left(t + \frac{1}{2}\right)}$$

TERCERA PARTE

(1) (a) $mg = k\Delta x \Rightarrow k = \frac{mg}{\Delta x} = \frac{2 \cdot g}{5/100} = 40g\ N/m$

$$\boxed{k \cong 400\ N/m}, \quad \text{if } g \approx 10\ m/sec^2$$

Tenemos:

$my'' + ky = 0$, since there is no viscosity.

$\Rightarrow y'' = -\omega^2 y, \quad \omega^2 = \frac{k}{m} \approx \frac{400}{2} = 200 = 2 \cdot 10^2$

$\Rightarrow \omega = 10 \cdot \sqrt{2}\ 1/sec. \cong 14 (1/sec.)$

$$\Rightarrow \boxed{y(t) = A\cos(\omega t) + B\sin(\omega t)}$$

$y'(t) = -\omega A\sin(\omega t) + \omega B\cos(\omega t)$

$-\frac{15}{100}m = y(0) = A \cdot 0 + B \cdot 0 \Rightarrow$

$0 = y'(0) = -\omega A \cdot 0 + \omega \cdot B \Rightarrow$

$$\boxed{A = \frac{-15}{100} \quad B = 0}$$

$$\boxed{y(t) = \frac{-15}{100}\cos(10\sqrt{2}t)}$$

(b) $M = \sqrt{A^2 + B^2} = \sqrt{\left(-\frac{15}{100}\right)^2 + 0^2} = \frac{15}{100}$

$\tan\phi = \frac{A}{B} = \text{"}\frac{-15/100}{0}\text{"} = \text{"}-\infty\text{"} \Rightarrow \phi = -\frac{\pi}{2}$

$$\Rightarrow \boxed{y(t) = \frac{15}{100}\sin\left(10\sqrt{2}t - \frac{\pi}{2}\right)}$$

(c) $y(T) = 0 \Rightarrow 10\sqrt{2}T - \frac{\pi}{2} = n\pi \Rightarrow \boxed{T = \left(\frac{2n+1}{2}\pi\right)\frac{1}{10\sqrt{2}}}$

$n = 0, 1, 2, 3, \ldots$

$\geq 1 \approx$

② (a) $m = \frac{4}{10}$ kg, $k = 4$ N/m.

$$m y'' + \beta y' + k y = 0$$

$$\frac{2}{5} y'' + \beta y' + 4y = 0$$

$$\frac{2}{5} r^2 + \beta r + 4 = 0$$

Critically damped: $\beta^2 - 4\left(\frac{2}{5}\right) \cdot 4 = 0 \Rightarrow \beta^2 = \frac{32}{5}$

$$\Rightarrow \beta^2 = \frac{2^5}{5} \Rightarrow \beta = 2^2\sqrt{\frac{2}{5}} \Rightarrow \boxed{\beta = 4\sqrt{2/5}}$$

[0 bonus: $\beta^2 = \frac{64}{10} \Rightarrow \boxed{\beta = \frac{8}{\sqrt{10}}}$

(b) Hence:

$$\frac{2}{5} r^2 + 4\sqrt{\frac{2}{5}}\, r + 4 = 0$$

$$\left(\sqrt{\frac{2}{5}}\, r + 2\right)^2 = 0 \Rightarrow r = -2\sqrt{\frac{5}{2}}$$

$$\Rightarrow r = -\sqrt{\frac{2^3 \cdot 5}{2}} = -\sqrt{10}. \qquad \boxed{r = -\sqrt{10}}$$

$$\Rightarrow \boxed{y(t) = (C_1 t + C_0)\, e^{-\sqrt{10}\,t}}$$

Now $\frac{1}{2} = y(0) = (0 + C_0) \cdot 1 \Rightarrow \boxed{C_0 = \frac{1}{2}}$

$$y'(t) = C_1 e^{-\sqrt{10}\,t} + (C_1 t + C_0)(-\sqrt{10})\, e^{-\sqrt{10}\,t}.$$

$$0 = y'(0) = C_1 + (-\sqrt{10})\, C_0 = C_1 - \frac{\sqrt{10}}{2} \Rightarrow \boxed{C_1 = \frac{\sqrt{10}}{2}}$$

$$\Rightarrow \boxed{y(t) = \left(\frac{\sqrt{10}\,t + 1}{10}\right) e^{-\sqrt{10}\,t}}$$

(c) $y(T) = 0 \Rightarrow \left(\frac{\sqrt{10}\,T + 1}{10}\right) e^{-\sqrt{10}\,T} = 0 \Rightarrow \boxed{T = -\frac{1}{\sqrt{10}} \text{ sec}}$

Never ~~passes~~