Tarea #1 Series, Transformadas y Ecuaciones Diferenciales.

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Problemas de: LINEAR ALGEBRA AND ITS APPLICATIONS Gilbert Strang

Chapter 5

(1) Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$$

Verify that the trace equals the sum of eigenvalues and the determinant equals the product of the eigenvalues.

Characteristic eq'n: $\det(A - \lambda I) = 0$.

$$A - \lambda I = \begin{pmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{pmatrix} \Rightarrow \det\begin{pmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{pmatrix} = 0$$

$$(1-\lambda)(4-\lambda) + 2 = 0$$

$$(\lambda-1)(\lambda-4) + 2 = 0$$

$$\lambda^2 - \lambda - 4\lambda + 4 + 2 = 0$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$\boxed{\begin{matrix}\lambda_1 = 2 \\ \lambda_2 = 3\end{matrix}}$ are the eigenvalues $\Leftarrow (\lambda - 3)(\lambda - 2) = 0$

For $\lambda_1 = 2$: $(A - \lambda_1)\vec{x}_1 = \vec{0} \Rightarrow \begin{pmatrix} 1-2 & -1 \\ 2 & 4-2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -x_1 - y_1 = 0 \Rightarrow x_1 = -y_1$$

Choose $y_1 = -1 \Rightarrow \boxed{\vec{x}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$

For $\lambda_2 = 3$: $\begin{pmatrix} 1-3 & -1 \\ 2 & 4-3 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -2x_2 - y_2 = 0$

$$\Rightarrow y_2 = -2x_2$$

Choose $x_2 = 1 \Rightarrow y_2 = -2 \Rightarrow \boxed{\vec{x}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}}$

$\mathrm{Tr}\, A = 1 + 4 = 5$

and $\det A = 1\cdot 4 - (2)(-1) = 6.$

Now $\lambda_1 + \lambda_2 = 2 + 3 = 5 \Rightarrow \boxed{\mathrm{Tr}\, A = \lambda_1 + \lambda_2}$

$\lambda_1 \lambda_2 = 2 \cdot 3 = 6 \Rightarrow \boxed{\det A = \lambda_1 \lambda_2}$

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(2) With the same matrix, solve the differential eq'n $\frac{d\vec{u}}{dt} = A\vec{u}$, with initial condition $\vec{u}(0) = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$.

What are the pure exponential solutions?

The two exponential solutions are:

$$\vec{u}_1(t) = e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \vec{u}_2(t) = e^{3t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

The general solution is,

$$\vec{u}(t) = c_1 e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Under the initial conditions.

$$c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$$

i.e. $\begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \end{pmatrix} \Rightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{-1}\begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 6 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 6 \end{pmatrix} = \begin{pmatrix} 1\cdot 6 \\ -1\cdot 6 \end{pmatrix} = \begin{pmatrix} 6 \\ -6 \end{pmatrix}$$

$$\Rightarrow \boxed{\vec{u}(t) = 6e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} - 6e^{3t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}}$$

③ If we shift A to $A-7I$, what are the eigenvalues and eigenvectors and how are related to those of A?

$$B = A - 7I = \begin{pmatrix} -6 & -1 \\ 2 & -3 \end{pmatrix}.$$

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Let us call β_1, β_2 the eigenvalues of B, and $\bar{y}_1, \bar{y}_2$ its corresponding eigenvectors. Now:

$$B - \beta I = \begin{pmatrix} -6-\beta & -1 \\ 2 & -3-\beta \end{pmatrix}$$

Characteristic equation $\det(B - \beta I) = 0$

i.e. $(\beta+6)(\beta+3) + 2 = 0 \Rightarrow \beta^2 + 9\beta + 18 + 2 = 0$

$\Rightarrow \beta^2 + 9\beta + 20 = 0 \Rightarrow (\beta+5)(\beta+4) = 0$

$$\Rightarrow \boxed{\begin{matrix} \beta_1 = -5 \\ \beta_2 = -4 \end{matrix}}$$

Eigenvectors For $\beta_1 = -5$:

$$\begin{pmatrix} -6+5 & -1 \\ 2 & -3+5 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} -x_1 - x_2 = 0 \\ 2x_1 + 2x_2 = 0 \end{matrix} \Rightarrow x_2 = -x_1$$

Choose $x_1 = 1 \Rightarrow \boxed{\bar{y}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$

For $\beta_2 = -4$

$$\begin{pmatrix} -6+4 & -1 \\ 2 & -3+4 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} -2x_1 - y_2 = 0 \\ 2x_1 + y_2 = 0 \end{matrix} \Rightarrow y_2 = -2x_1$$

Choose $x_1 = 1 \Rightarrow \boxed{\bar{y}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}}$

Then.
$$\beta_1 = \lambda_1 - 7 = 2 - 7 = -5$$
$$\beta_2 = \lambda_2 - 7 = 3 - 7 = -4$$

i.e. the eigenvalues should <u>be shifted by 7</u>.

The eigenvectors, $\bar{x}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \bar{y}_1$; $\bar{x}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \vec{y}_2$, <u>remain the same</u> !!!

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(4) Solve $\dfrac{du}{dt} = Pu$, P is a projection: $P^2 = P$.

$$P = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}, \text{ with initial conditions } u(0) = \begin{pmatrix} 5 \\ 3 \end{pmatrix}.$$

$$P - \lambda I = \begin{pmatrix} 1/2 - \lambda & 1/2 \\ 1/2 & 1/2 - \lambda \end{pmatrix}$$

Characteristic eq'n $\det(P - \lambda I) = 0$.

$$\Rightarrow (1/2 - \lambda)^2 - 1/4 = 0 \Rightarrow \lambda^2 - \lambda + \frac{1}{4} - \frac{1}{4} = 0.$$

$$\Rightarrow \lambda^2 - \lambda = 0 \Rightarrow \lambda(\lambda - 1) = 0 \Rightarrow \boxed{\begin{matrix} \lambda_1 = 0 \\ \lambda_2 = 0 \end{matrix}}$$

Eigenvalues

<u>Eigenvectors for $\lambda_1 = 0$</u>

$$\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \frac{x_1}{2} + \frac{y_1}{2} = 0 \Rightarrow \boxed{\vec{x}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

<u>For $\lambda_2 = 1$.</u>

$$\begin{pmatrix} 1/2 - 1 & 1/2 \\ 1/2 & 1/2 - 1 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-\frac{1}{2} x_2 + \frac{1}{2} y_2 = 0 \Rightarrow \boxed{\bar{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

The general solution is:

$$\vec{u}(t) = c_1 e^{2t}\begin{pmatrix}1\\-1\end{pmatrix} + c_2 e^{t}\begin{pmatrix}1\\1\end{pmatrix}$$

$$\Rightarrow \quad c_1\begin{pmatrix}1\\-1\end{pmatrix} + c_2\begin{pmatrix}1\\1\end{pmatrix} = \vec{u}(0) = \begin{pmatrix}5\\3\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}1 & 1\\-1 & 1\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}5\\3\end{pmatrix} \Rightarrow \begin{pmatrix}c_1\\c_2\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & -1\\1 & 1\end{pmatrix}\begin{pmatrix}5\\3\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}c_1\\c_2\end{pmatrix} = \frac{1}{2}\begin{pmatrix}5-3\\5+3\end{pmatrix} = \frac{1}{2}\begin{pmatrix}2\\8\end{pmatrix} = \begin{pmatrix}1\\4\end{pmatrix}$$

Solution to the I.V.P. $\boxed{\vec{u}(t) = \begin{pmatrix}1\\-1\end{pmatrix} + 4e^{t}\begin{pmatrix}1\\1\end{pmatrix}}$

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(7) Suppose A is eigenvalue λ, and $\vec{x}$ eigenvector of A.

(a) Show that this same eigenvector $\vec{x}$ is eigenvector of $B = A - 7I$. Find the eigenvalue. This shows problem 3.

We have: $\vec{x}$ eivectore of A.

$$B\vec{x} = (A - 7I)\vec{x} = A\vec{x} - 7I\vec{x} = \lambda\vec{x} - 7\vec{x} = (\lambda - 7)\vec{x}$$

i.e. $B\vec{x} = (\lambda - 7)\vec{x}$

i.e. $\vec{x}$ is eigenvector $\vec{x}$ of B, with eigenvalue $\boxed{\beta = \lambda - 7.}$

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(12) Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix}3 & 4\\4 & -3\end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix}a & b\\b & a\end{pmatrix}$$

(a) For $A = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} \Rightarrow A - \lambda I = \begin{pmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{pmatrix}$

Characteristic equation: $\det(A - \lambda I) = 0$

$$\Rightarrow (3-\lambda)(-3-\lambda) - 16 = 0$$
$$(\lambda - 3)(\lambda + 3) - 16 = 0$$
$$\lambda^2 - 9 - 16 = 0 \Rightarrow \lambda^2 = 25 \Rightarrow \boxed{\lambda_{1,2} = \pm 5}$$

For $\lambda_1 = 5$ $\quad (A - \lambda_1 I)\vec{x}_1 = \vec{0} \Rightarrow \begin{pmatrix} 3-5 & 4 \\ 4 & -3-5 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} -2 & 4 \\ 4 & -8 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -2x_1 + 4y_1 = 0 \quad \vec{x}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

For $\lambda_2 = -5$:

$$\begin{pmatrix} 3+5 & 4 \\ 4 & -3+5 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 8x_2 + 4y_2 = 0$$
$$\Rightarrow \vec{x}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Then, A has eigenvalue $\boxed{\lambda_1 = 5}$ with corresponding eigenvector $\boxed{\vec{x}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}}$

and a 2nd eigenvalue $\boxed{\lambda_2 = -5}$ with corresponding eigenvalue $\boxed{\vec{x}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}}$

(b) $A = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \Rightarrow A - \lambda I = \begin{pmatrix} a-\lambda & b \\ b & a-\lambda \end{pmatrix}$

Characteristic equation: $\det(A - \lambda I) = 0$

$$\Rightarrow (a-\lambda)^2 - b^2 = 0 \Rightarrow \lambda^2 - 2a\lambda + a^2 - b^2 = 0$$

$$\lambda_{1,2} = \frac{2a \pm \sqrt{4a^2 - 4(a^2 - b^2)}}{2} \Rightarrow \lambda_{1,2} = \frac{2a \pm \sqrt{4b^2}}{2}$$

$$\lambda_{1,2} = \frac{2a \pm 2|b|}{2} = a \pm |b|$$

$$\Rightarrow \boxed{\lambda_{1,2} = a \pm b}$$

For $\lambda_1 = a+b$.

$$(A - \lambda_1 I)\vec{x}_1 = \vec{0} \Rightarrow \begin{pmatrix} a-\lambda_1 & b \\ b & a-\lambda_1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} -b & b \\ b & -b \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \vec{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

For $\lambda_2 = a-b$

$$(A - \lambda_2 I)\vec{x}_2 = \vec{0} \Rightarrow \begin{pmatrix} b & b \\ b & b \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Hence, the eigenpairs are:

$$\boxed{\begin{matrix} \lambda_1 = a+b \\ \vec{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{matrix}} \quad \boxed{\begin{matrix} \lambda_2 = a-b \\ \vec{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{matrix}}$$

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(16) If A is the 4×4 matrix with 1's, find the eigenvalues and the determinant of $A - I$:

Since the columns of A are linearly dependent, at least one eigenvalue is 0.

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}; \quad A - \lambda I = \begin{pmatrix} 1-\lambda & 1 & 1 & 1 \\ 1 & 1-\lambda & 1 & 1 \\ 1 & 1 & 1-\lambda & 1 \\ 1 & 1 & 1 & 1-\lambda \end{pmatrix}$$

Characteristic equation:

$$\det(A - \lambda I) = 0$$

$$(1-\lambda)\det\begin{pmatrix}1-\lambda & 1 & 1\\ 1 & 1-\lambda & 1\\ 1 & 1 & 1-\lambda\end{pmatrix} - 1\cdot\det\begin{pmatrix}1 & 1 & 1\\ 1 & 1-\lambda & 1\\ 1 & 1 & 1-\lambda\end{pmatrix}$$

$$+ 1\cdot\det\begin{pmatrix}1 & 1-\lambda & 1\\ 1 & 1 & 1\\ 1 & 1 & 1-\lambda\end{pmatrix} - \det\begin{pmatrix}1 & 1-\lambda & 1\\ 1 & 1 & 1-\lambda\\ 1 & 1 & 1\end{pmatrix}$$

$$= (1-\lambda)\det\begin{pmatrix}1-\lambda & 1 & 1\\ 1 & 1-\lambda & 1\\ 1 & 1 & 1-\lambda\end{pmatrix} - \det\begin{pmatrix}1 & 1 & 1\\ 1 & 1-\lambda & 1\\ 1 & 1 & 1-\lambda\end{pmatrix} - \det\begin{pmatrix}1 & 1 & 1\\ 1 & 1-\lambda & 1\\ 1 & 1 & 1-\lambda\end{pmatrix}$$

$$- \det\begin{pmatrix}1 & 1 & 1\\ 1 & 1-\lambda & 1\\ 1 & 1 & 1-\lambda\end{pmatrix}$$

$$= (1-\lambda)\overbrace{\det\begin{pmatrix}1-\lambda & 1 & 1\\ 1 & 1-\lambda & 1\\ 1 & 1 & 1-\lambda\end{pmatrix}}^{D_1 \equiv} - 3\underbrace{\det\begin{pmatrix}1 & 1 & 1\\ 1 & 1-\lambda & 1\\ 1 & 1 & 1-\lambda\end{pmatrix}}_{D_2 =}$$

Now, on the one hand:

$$D_1 = \det\begin{pmatrix}1-\lambda & 1 & 1\\ 1 & 1-\lambda & 1\\ 1 & 1 & 1-\lambda\end{pmatrix} = (1-\lambda)\det\begin{pmatrix}1-\lambda & 1\\ 1 & 1-\lambda\end{pmatrix} - 1\cdot\det\begin{pmatrix}1 & 1\\ 1 & 1-\lambda\end{pmatrix}$$

$$+ 1\cdot\det\begin{pmatrix}1 & 1-\lambda\\ 1 & 1\end{pmatrix}$$

$$= (1-\lambda)\det\begin{pmatrix}1-\lambda & 1\\ 1 & 1-\lambda\end{pmatrix} - 2\det\begin{pmatrix}1 & 1\\ 1 & 1-\lambda\end{pmatrix}$$

$$= (1-\lambda)\left[(1-\lambda)^2 - 1\right] - 2\left[(1-\lambda) - 1\right]$$

$$= (1-\lambda)\left[(1-\lambda)^2 - 1\right] + 2\lambda$$

$$= (1-\lambda)\left[1 - 2\lambda + \lambda^2 - 1\right] + 2\lambda$$

$$= (1-\lambda)(\lambda^2 - 2\lambda) + 2\lambda$$

$$= \lambda(1-\lambda)(\lambda-2) + 2\lambda$$

$$= \lambda\left[(1-\lambda)(\lambda-2) + 2\right]$$

$$= \lambda\left[-\lambda^2 + \lambda + 2\lambda - 2 + 2\right]$$

$$= \lambda\left[-\lambda^2 + 3\lambda\right] = -\lambda^2[\lambda-3].$$

And on the other hand:

$$D_2 = \det\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{pmatrix} = 1\cdot\det\begin{pmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} - 1\det\begin{pmatrix} 1 & 1 \\ 1 & 1-\lambda \end{pmatrix} + 1\cdot\det\begin{pmatrix} 1 & 1-\lambda \\ 1 & 1 \end{pmatrix}$$

$$= \det\begin{pmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} - 2\det\begin{pmatrix} 1 & 1 \\ 1 & 1-\lambda \end{pmatrix}$$

$$= \left[(1-\lambda)^2 - 1\right] - 2\left[(1-\lambda) - 1\right] = \left[(1-\lambda)^2 - 1\right] - 2[-\lambda]$$

$$= \left[\lambda^2 - 2\lambda\right] + 2\lambda = \lambda^2$$

Hence:

$$\det(A - \lambda I) = (1-\lambda)D_1 - 3D_2$$

$$= (1-\lambda)\left[-\lambda^2(\lambda-3)\right] - 3\lambda^2$$

$$= -(1-\lambda)\lambda^2(\lambda-3) - 3\lambda^2$$

$$= \lambda^2\left[(\lambda-1)(\lambda-3) - 3\right]$$

$$= \lambda^2\left[\lambda^2 - 4\lambda + 3 - 3\right]$$

$$\det(A - \lambda I) = \lambda^2\left[\lambda^2 - 4\lambda\right]$$

$$\boxed{\det(A - \lambda I) = \lambda^3[\lambda - 4]}$$

The eigenvalues are $\lambda_1 = 0$
$\lambda_2 = 0$
$\lambda_3 = 0$
$\lambda_4 = 4$

$\det(A - I) = 1^3[1-$
$= -3$

where we put $\lambda = 1$.

20. Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \quad \text{and} \quad A + I = \begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix}$$

$A+I$ has the <u>same</u> eigenvectors as A

Its eigenvalues are <u>shifted</u> by 1.

$$A - \lambda I = \begin{pmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{pmatrix}$$

$\det(A - \lambda I) = 0$

$\Rightarrow (\lambda-1)(\lambda-3) - 8 = 0 \Rightarrow \lambda^2 - 4\lambda + 3 - 8 = 0$

$\Rightarrow \lambda^2 - 4\lambda - 5 = 0 \Rightarrow (\lambda - 5)(\lambda + 1) = 0$ $\boxed{\begin{matrix} \lambda_1 = -1 \\ \lambda_2 = 5 \end{matrix}}$

The eigenvalues of $A+I$; denote them by β:

$\det((A+I) - \beta I) = 0 \Rightarrow \det(A - (\beta - 1)I) = 0$

$\Rightarrow \lambda = \beta - 1$ are the eigenvalues of A

$\Rightarrow \beta = \lambda + 1$ are the eigenvalues of $A+I$: <u>shifted by 1</u>.

The eigenvalues of $A + I$ are $\boxed{\begin{matrix} \beta_1 = \lambda_1 + 1 = 0 \\ \beta_2 = \lambda_2 + 1 = 6 \end{matrix}}$

(See problem 7(a)).

(22) Compute the eigenvalues and eigenvectors of A and A^2:

$$A = \begin{pmatrix} -1 & 3 \\ 2 & 0 \end{pmatrix} ; \quad A^2 = \begin{pmatrix} 7 & -3 \\ -2 & 6 \end{pmatrix}.$$

A^2 has the same eigenvectors as A.

when A has eigenvalues λ_1 y λ_2, A^2 has eigenvalues λ_1^2 y λ_2^2.

If λ and $\vec{x}$ is eigenpair of A:

$$A\vec{x} = \lambda\vec{x}.$$

Then: $A(A\vec{x}) = A(\lambda\vec{x}) \Rightarrow A^2\vec{x} = \lambda(A\vec{x}) \Rightarrow A^2\vec{x} = \lambda(\lambda\vec{x})$

$\Rightarrow A^2\vec{x} = \lambda^2\vec{x}$. Then, $\vec{x}$ is eigenvector of A^2, with the corresponding eigenvalue λ^2.

$$A - \lambda I = \begin{pmatrix} -1-\lambda & 3 \\ 2 & -\lambda \end{pmatrix}$$

$$\det(A - \lambda I) = 0$$

$$(-1-\lambda)(-\lambda) - 6 = 0 \longrightarrow \lambda^2 + \lambda - 6 = 0.$$

$$(\lambda+3)(\lambda-2) = 0.$$

$$\boxed{\begin{matrix} \lambda_1 = 2 \\ \lambda_2 = -3 \end{matrix}}$$

Eigenvectors

For $\lambda_1 = 2$. $\begin{pmatrix} -1-2 & 3 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad -3x_1 + 3y_1 = 0$

$$\vec{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = -3$ $\begin{pmatrix} -1+3 & 3 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$; $\quad 2x_2 + 3y_2 = 0$

$$\vec{x}_2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

Eigenpairs for A

$\lambda_1 = 2$; $\vec{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = -3$, $\vec{x}_2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

Eigenpairs for A^2

$\lambda_1 = 4$; $\vec{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = 9$; $\vec{x}_2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

(30) Choose the second row of $A = \begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix}$ so that A eivalues 4 and 7.

We could compute $\det(A - \lambda I) = 0$, and solve for a and b. We could also do:

$$28 = \lambda_1 \lambda_2 = \det A = -a \Rightarrow a = -28$$

and

$$11 = \lambda_1 + \lambda_2 = tr(A) = 0 + b \Rightarrow b = 11$$

$$\Rightarrow A = \begin{pmatrix} 0 & 1 \\ -28 & 11 \end{pmatrix}.$$

(31) Choose a, b, c, so that $\det(A - \lambda I) = 9\lambda - \lambda^3$.

i.e. $\lambda(9 - \lambda^2) = \lambda(3-\lambda)(3+\lambda)$.

i.e., the eivalues are $-3, 0, 3$.

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{pmatrix}.$$

$$\det(A) = (-1)\det\begin{pmatrix} 0 & 1 \\ a & c \end{pmatrix} = -a$$

$$tr(A) = c.$$

Then: $-a = \lambda_1 \lambda_2 \lambda_3 = 0 \Rightarrow \boxed{a = 0}$

$c = \lambda_1 + \lambda_2 + \lambda_3 = 0 \Rightarrow \boxed{c = 0.}$

Now: $\det(A - \lambda I) = \det\begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & b & -\lambda \end{pmatrix} =$

$$= (-\lambda)\det\begin{pmatrix} -\lambda & 1 \\ b & -\lambda \end{pmatrix} - 1\begin{vmatrix} 0 & 1 \\ 0 & -\lambda \end{vmatrix} = (-\lambda)(\lambda^2 - b) = \lambda(b - \lambda^2)$$

$$\Rightarrow \boxed{b = 9}$$

Problemas más elaborados.

(8) Show that the determinant equals the product of eivals by imagining that the characteristical polynomial is factored into:

$$\det(A-\lambda I) = (\lambda_1-\lambda)(\lambda_2-\lambda)\cdots(\lambda_n-\lambda).$$

and making a clever choice of λ.

Choos $\lambda=0$; then:

$$\det A = \lambda_1\lambda_2\cdots\lambda_n.$$

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(5) Find the eigenvalues and eigenvectors of:

$$A = \begin{pmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix}.$$

Check that $\lambda_1+\lambda_2+\lambda_3$ is the trace and $\lambda_1\lambda_2\lambda_3$ its det.

$$\det(A-\lambda I) = \det\begin{pmatrix} 3-\lambda & 4 & 2 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & -\lambda \end{pmatrix}$$

$$= (3-\lambda)\det\begin{pmatrix} 1-\lambda & 2 \\ 0 & -\lambda \end{pmatrix} - 4\det\begin{pmatrix} 0 & 2 \\ 0 & -\lambda \end{pmatrix} + 2\det\begin{pmatrix} 0 & 1-\lambda \\ 0 & 0 \end{pmatrix}$$

$$= (3-\lambda)(1-\lambda)(-\lambda) - 0 + 0$$

$$= -\lambda(\lambda-1)(\lambda-3)$$

$$\boxed{\begin{aligned} \lambda_1 &= 0 \\ \lambda_2 &= 1 \\ \lambda_3 &= 3 \end{aligned}}$$

For $\lambda_1 = 0$

$$\begin{pmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \vec{X}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{array}{r} 3x_1 + 4y_1 + 2z_1 = 0 \\ y_1 + 2z_1 = 0 \end{array}\right\} \Rightarrow 3x_1 + 4y_1 - y_1 = 0.$$

$$\Rightarrow 3x_1 - 3y_1 = 0$$

$\Rightarrow x_1 = y_1$ and z_1 arbitrary

$$\boxed{\vec{X}_1 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}} \Leftarrow \left\{\begin{array}{l} \Rightarrow x_1 = -2z_1 \\ y_1 = z_1 \\ z_1 \text{ arbitrary.} \end{array}\right.$$

For $\lambda_2 = 1$:

$$\begin{pmatrix} 2 & 4 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{array}{r} 2x_2 + 4y_2 + 2z_2 = 0 \\ 2z_2 = 0 \\ -z_2 = 0 \end{array}$$

$$\Rightarrow z_2 = 0 \Rightarrow 2x_2 + 4y_2 \Rightarrow x_2 = -2y_2$$

$$\Rightarrow \boxed{\vec{X}_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}}$$

For $\lambda_3 = 3$

$$\begin{pmatrix} 0 & 4 & 2 \\ 0 & -2 & 2 \\ 0 & 0 & -3 \end{pmatrix}\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{array}{r} 4y_3 + 2z_3 = 0 \\ -2y_3 + 2z_3 = 0 \\ -3z_3 = 0 \end{array}$$

$\Rightarrow z_3 = 0 \Rightarrow -2y_3 = 0 \Rightarrow y_3 = 0$ and x_3 arbitrary:

$$\boxed{\vec{X}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$$

Eigenpairs of A:

$$\boxed{\lambda_1 = 0\,;\ \vec{x}_1 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}} \quad \boxed{\lambda_2 = 1\,;\ \vec{x}_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}}$$

$$\boxed{\lambda_3 = 3,\quad \vec{x}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$$

For the matrix $B = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix}$

$$\det(B - \lambda I) = \det\begin{pmatrix} -\lambda & 0 & 2 \\ 0 & 2-\lambda & 0 \\ 2 & 0 & -\lambda \end{pmatrix} = (-\lambda)\det\begin{pmatrix} 2-\lambda & 0 \\ 0 & -\lambda \end{pmatrix} + 2\det\begin{pmatrix} 0 & 2-\lambda \\ 2 & 0 \end{pmatrix}$$

$$= (-\lambda)(2-\lambda)(-\lambda) + 2(-1)2(2-\lambda)$$

$$= \lambda^2(2-\lambda) - 4(2-\lambda) = (\lambda^2-4)(2-\lambda) = -(\lambda-2)^2(\lambda+2).$$

$$\Rightarrow \text{Eigenvalues: } \boxed{\begin{aligned} \lambda_1 &= 2 \\ \lambda_2 &= 2 \\ \lambda_3 &= -2 \end{aligned}}$$

Eigenvectors.

$\lambda_3 = -2$ $\quad (B - \lambda_3 I)\vec{x}_3 = 0 \quad \begin{pmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 2 \end{pmatrix}\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left.\begin{aligned} 2x_3 + 2z_3 &= 0 \\ 4y_3 &= 0 \end{aligned}\right\} \Rightarrow y_3 = 0 \text{ and } x_3 = -z_3$$

$$\Rightarrow \vec{x}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

For: $\underline{\lambda_1 = 2}$

$$\begin{pmatrix} -2 & 0 & +2 \\ 0 & 0 & 0 \\ 2 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-2x_1 + 2z_1 = 0 \qquad x_1 = z_1$$

$$0 = 0 \quad \Rightarrow \quad z_1 \text{ and } y_1 \text{ arbitrary.}$$

$$2x_1 - 2z_1 = 0$$

$$\Rightarrow \quad \bar{X}_1 = \begin{pmatrix} 1 \\ y_1 \\ 1 \end{pmatrix} \qquad \text{Take: } y_1 = 1$$

$$\boxed{\bar{X}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$$

The same applies to $\lambda_2 = 2$:

$$\bar{X}_2 = \begin{pmatrix} 1 \\ y_2 \\ 1 \end{pmatrix} \qquad \text{Take } y_2 = 0 \qquad \boxed{\bar{X}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}$$

$\boxed{\text{For } A:}$

$$\det A = \det \begin{pmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} = 0 \text{, since } 3^{rd} \text{ row is } (0,0,0).$$

$$\operatorname{tr} A = 3 + 1 = 4$$

Now $\lambda_1 \lambda_2 \lambda_3 = 0 \cdot 1 \cdot 3 = 0 = \det A$ ✓

$$\lambda_1 + \lambda_2 + \lambda_3 = 4 = \operatorname{tr}(A) \ ✓$$

$\boxed{\text{For } B}$ $\det B = \det \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix} = 2 \begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix} = -8.$

$$\operatorname{tr} B = 0 + 2 + 0 = 2$$

$$\lambda_1 \lambda_2 \lambda_3 = 2 \cdot 2 \cdot (-2) = -8 = \det B$$

$$\lambda_1 + \lambda_2 + \lambda_3 = 2 + 2 - 2 = 2 = \operatorname{tr} B$$

1) Show that the trace equals the sum of eigenvalues in two steps:

1st. Find the coefficient of $(-\lambda)^{n-1}$ on the RHS of

$$\det(A-\lambda I) = (\lambda_1-\lambda)(\lambda_2-\lambda)\cdots(\lambda_{n-1}-\lambda)(\lambda_n-\lambda),$$

$$(*) \quad = (-\lambda)^n + (-\lambda)^{n-1}(\lambda_n+\lambda_{n-1}+\cdots+\lambda_2+\lambda_1) + (-\lambda)^{n-2}\cdots$$

2nd. Find the terms that involve $(-\lambda)^{n-1}$ in:

$$\det(A-\lambda I) = \det\begin{pmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{pmatrix} =$$

$$= (a_{11}-\lambda)\underbrace{\det\begin{pmatrix} a_{22}-\lambda & \cdots & a_{2n} \\ a_{n2} & & a_{nn}-\lambda \end{pmatrix}}_{} - a_{12}\underbrace{\det\begin{pmatrix} a_{21} & a_{23} & \cdots \\ a_{31} & a_{33}-\lambda & \\ a_{n1} & \cdots & \end{pmatrix}}_{\text{Order } (-\lambda)^{n-2}+\cdots}$$

$$= (a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{n-1\,n-1}-\lambda)(a_{nn}-\lambda) + \cdots \text{ Order } (-\lambda)^{n-2}$$

$$(**) \quad = (-\lambda)^n + (-\lambda)^{n-1}(a_{11}+a_{22}+\cdots+a_{nn}) + \cdots \text{ Order } (-\lambda)^{n-2} + \cdots$$

Compare eq'n $(*)$ and $(**)$:

$$a_{11}+a_{22}+\cdots+a_{nn} = \lambda_1+\lambda_2+\cdots+\lambda_n.$$

$$\boxed{Tr(A) = \lambda_1+\lambda_2+\cdots+\lambda_n}$$

(24) What do you do to $A\vec{x} = \lambda\vec{x}$ in order to prove the following statements?

(a) λ^2 is eigenvalue of A^2

Apply A to $A\vec{x} = \lambda\vec{x}$ on both sides:

$$
\begin{aligned}
A^2\vec{x} &= A(\lambda\vec{x}) = \\
&= \lambda(A\vec{x}) \\
&= \lambda(\lambda\vec{x}) \\
&= \lambda^2\vec{x}
\end{aligned}
$$

(b) λ^{-1} is an ~~eigenvector~~ eigenvalue of A^{-1}.

Multiply from the left by A^{-1} on both sides of $A\vec{x} = \lambda\vec{x}$.

$$A^{-1}A\vec{x} = \lambda(A^{-1}\vec{x})$$

$$\Rightarrow \frac{1}{\lambda}\vec{x} = A^{-1}\vec{x} \quad \Rightarrow \quad A^{-1}\vec{x} = \lambda^{-1}\vec{x}.$$

(c) $\lambda + 1$ is eigenvalue of $A + I$.

Add $\vec{x}$ on both sides: of $A\vec{x} = \lambda\vec{x}$.

$$A\vec{x} = \lambda\vec{x}.$$

$$A\vec{x} + \vec{x} = \lambda\vec{x} + \vec{x}$$

$$\boxed{(A + I)\vec{x} = (\lambda + 1)\vec{x}.}$$