UAM-Azcapotzalco Tarea #2. Invierno 2017

Series, Transformadas y Ecuaciones Diferenciales

Zill-Cullen. Secciones 8.1. Ejercicios 1, 7, 11, 17, 19, 21, 26

8.2 Ejercicios 1, 13, 14.

(1) $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -5 \\ 4 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

(7) $\frac{dx}{dt} = 3x - 4y + e^t$

$\frac{dy}{dt} = 4x - 7y - e^t.$

(11) $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^{-5t} \\ 2e^{-5t} \end{pmatrix}$; $\frac{dx}{dt} = -5e^{-5t}$

$\frac{dy}{dt} = -10e^{-5t}.$

Now:

$3x - 4y + e^t = 3(e^{-5t}) - 4(2e^{-5t}) = (3-8)e^{-5t} = -5e^{-5t}$ ✓

$4x - 7y = 4(e^{-5t}) - 7(2e^{-5t}) = (4-14)e^{-5t} = -10e^{-5t}$

(17) $\vec{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t}$; $\vec{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-6t}$

$\det(\vec{x}_1, \vec{x}_2) = \det\begin{pmatrix} e^{-2t} & e^{-6t} \\ e^{-2t} & -e^{-6t} \end{pmatrix} = -e^{-8t} - e^{-8t} = -2e^{-8t}.$

It is a fundamental set since $\det(\vec{x}_1, \vec{x}_2) \neq 0$.

(19) $\vec{x}_1 = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$, $\vec{x}_2 = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$

$$\vec{x}_3 = \begin{pmatrix} 3 \\ -6 \\ 12 \end{pmatrix} + t\begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix}.$$

At $t=0$, $\vec{x}_1 = \vec{x}_2$, and they are linearly dependent.

For $t \geq 0$: $3\vec{x}_1 = \vec{x}_3$, i.e., they are linearly dependent.

Then, this is <u>not</u> a fundamental set of solutions.

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(21) $\frac{dx}{dt} = x + 4y + 2t - 7$. ; $\vec{x}_p(t) = \begin{pmatrix} 2t+5 \\ -t+1 \end{pmatrix}$.

$\frac{dy}{dt} = 3x + 2y - 4t - 18$

$\frac{dx_p}{dt} = 2$

$\frac{dy_p}{dt} = -1$

Also:

$$\begin{aligned} x+4y+2t-7 &= (2t+5) + 4(-t+1) + 2t - 7 \\ &= (2-4+2)t + (5+4-7) \\ &= 0 \cdot t + 2 = 2 \checkmark \end{aligned}$$

$$\begin{aligned} 3x+2y-4t-18 &= 3(2t+5) + 2(-t+1) - 4t - 18 \\ &= (6-2-4)t + (15+2-18) \\ &= 0 \cdot t + (-1) = -1 \checkmark \end{aligned}$$

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(26) Similarly, we just have to substitute and we're done.

Sect 8.2 Zill-Cullen

① Find the general solution of.

$$\frac{dx}{dt} = x + 2y \qquad A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$$

$$\frac{dy}{dt} = 4x + 3y$$

Characteristic equation: $\det(A - \lambda I) = 0$.

$$\det \begin{pmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{pmatrix} = (\lambda-1)(\lambda-3) - 8 = \lambda^2 - 4\lambda + 3 - 8 = 0$$

$$\Rightarrow \quad \lambda^2 - 4\lambda - 5 = 0; \quad (\lambda - 5)(\lambda+1) = 0 \quad \boxed{\begin{matrix} \lambda_1 = -1 \\ \lambda_2 = 5 \end{matrix}}$$

Eigenvectors.

For $\lambda_1 = -1$.

$$\begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x_1 + y_1 = 0$$

$$\Rightarrow \boxed{\vec{x}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

for $\lambda_2 = 5$

$$\begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \boxed{\vec{x}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}}.$$

The general solution is thus:

$$\boxed{\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{5t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}}$$

(13) Solve the I.V.P..

$$\vec{x}' = \begin{pmatrix} 1/2 & 0 \\ 1 & -1/2 \end{pmatrix} \vec{x} \quad , \quad \vec{x}(0) = \begin{pmatrix} 3 \\ 5 \end{pmatrix}.$$

Characteristic eq'n:

$$\det \begin{pmatrix} 1/2 - \lambda & 0 \\ 1 & -1/2 - \lambda \end{pmatrix} = 0 \Rightarrow (1/2 - \lambda)(-1)(\tfrac{1}{2} + \lambda) = 0$$

$$\Rightarrow \boxed{\begin{array}{l} \lambda_1 = -1/2 \\ \lambda_2 = 1/2 \end{array}}$$

Eigenvectors for $\lambda_1 = -1/2$

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad x_1 + 0 = 0 \qquad \Rightarrow \boxed{\vec{x}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

For $\lambda_2 = +1/2$

$$\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad x_2 - y_2 = 0 \qquad \boxed{\vec{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

The general solution is:

$$\vec{x}(t) = c_1 e^{-t/2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2 e^{t/2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

At $t = 0$

$$c_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{-1} \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} -(3-5) \\ -(-3) \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\Rightarrow \boxed{\vec{x}(t) = 2e^{-t/2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + 3e^{t/2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

(14) Solve the Initial Value problem.

$$\vec{x}' = \begin{pmatrix} 1 & 1 & 4 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix} \vec{x}, \qquad \vec{x}(0) = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}.$$

Characteristic equation

$$\det \begin{pmatrix} 1-\lambda & 1 & 4 \\ 0 & 2-\lambda & 0 \\ 1 & 1 & 1-\lambda \end{pmatrix} = 0.$$

$$(1-\lambda)\left|\begin{pmatrix} 2-\lambda & 0 \\ 1 & 1-\lambda \end{pmatrix}\right| - 1\cdot\left|\begin{pmatrix} 0 & 0 \\ 1 & 1-\lambda \end{pmatrix}\right| + 4\left|\begin{pmatrix} 0 & 2-\lambda \\ 1 & 1 \end{pmatrix}\right| = 0.$$

$$(1-\lambda)^2(2-\lambda) + 4(-1)(2-\lambda) = 0.$$

$$(2-\lambda)\left((1-\lambda)^2 - 4\right) = 0$$

$$(2-\lambda)(1-\lambda-2)(1-\lambda+2) = 0$$

$$(2-\lambda)(-1-\lambda)(3-\lambda) = 0 \Rightarrow$$

Eigenvalues

$\lambda_1 = -1$

$\lambda_2 = 2$

$\lambda_3 = 3$

Eigenvectors: For $\lambda_1 = -1$:

$$\begin{pmatrix} 2 & 1 & 4 \\ 0 & 3 & 0 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{aligned} 2x_1 + y_1 + 4z_1 &= 0 \\ 3y_1 &= 0 \\ x_1 + y_1 + 2z_1 &= 0 \end{aligned}$$

$$\Rightarrow y_1 = 0 \Rightarrow \begin{aligned} 2x_1 + 4z_1 &= 0 \\ x_1 + 2z_1 &= 0 \end{aligned} \Rightarrow x_1 = -2z_1$$

Choose $z_1 = 1$. $\boxed{\vec{x}_1 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}}$

For $\lambda_2 = 2$

$$\begin{pmatrix} -1 & 1 & 4 \\ 0 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \left.\begin{aligned} -x_2 + y_2 + 4z_2 &= 0 \\ x_2 + y_2 - z_2 &= 0 \end{aligned}\right\} \Rightarrow$$

$\Rightarrow \quad 2y_2 + 3z_2 = 0 \quad \Rightarrow y_2 = -\frac{3}{2} z_2$

and $\Rightarrow \quad -2x_2 + 5z_2 = 0 \quad \Rightarrow x_2 = \frac{5}{2} x_2$

Choose $z_2 = 2 \quad \Rightarrow \quad \boxed{\vec{x}_2 = \begin{pmatrix} +5 \\ -3 \\ 2 \end{pmatrix}}$

For $\lambda_3 = 3$.

$$\begin{pmatrix} -2 & 1 & 1 \\ 0 & -1 & 0 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{aligned} -2x_3 + y_2 + 4z_3 &= 0 \\ -y_3 &= 0 \\ x_3 + y_3 - 2z_3 &= 0 \end{aligned}$$

$\Rightarrow \quad y_3 = 0 \quad \Rightarrow \quad -2x_3 + 4z_3 = 0 \quad \Rightarrow \quad x_3 = 2z_3$

$x_3 - 2z_3 = 0$ Choose $z_3 = 1$ $\boxed{\vec{x}_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}}$

General solution is:

$$\vec{x}(t) = c_1 e^{-t} \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix} + c_3 e^{3t} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.$$

At $t=0$:

$$c_1 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix} + c_3 \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -2 & 5 & 2 \\ 0 & -3 & 0 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \frac{1}{12} \begin{pmatrix} -3 & -1 & 6 \\ 0 & -4 & 0 \\ 3 & 9 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 \\ -2 \\ 5 \end{pmatrix}$$

$$\Rightarrow \boxed{\vec{x}(t) = -\frac{1}{2} e^{-t} \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} - e^{2t} \begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix} + \frac{5}{2} e^{3t} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}}$$