EXAM #1. Differential Calculus. Thursday, February 11, 2016.
Universidad Autónoma Metropolitana – Azcapotzalco

(1). (a) $f(x) = \frac{1}{x}$.

Three steps:

(i) $f(x+h) - f(x) = \frac{1}{x+h} - \frac{1}{x} = \frac{x-(x+h)}{(x+h)\,x} = \frac{-h}{(x+h)(x)}$

(ii) $\frac{f(x+h)-f(x)}{h} = \frac{1}{h}\cdot\left(\frac{-h}{(x+h)\,x}\right) = \frac{-1}{(x+h)\,x}$.

(iii) $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{-1}{(x+h)\,x} = \frac{-1}{(x)x} = \frac{-1}{x^2}$

$\Rightarrow \quad f'(x) = -\frac{1}{x^2}$.

(b) Here, $x = 2$. $\Rightarrow f(2) = \frac{1}{2}$. Point $(2, \frac{1}{2})$.

Slope of tangent $= m = f'(2) = -\frac{1}{2^2} = -\frac{1}{4}$.

Straight line that passes through $(2, \frac{1}{2})$ with slope $-1/4$ has equation: $\boxed{y = -\frac{1}{4}(x-2) + \frac{1}{2}}$.

or $\boxed{y = -\frac{x}{4} + 1}$

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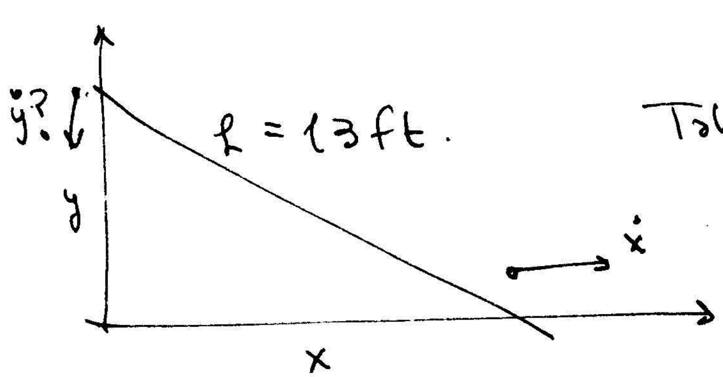
(2)



We have, by the Pythagorean theorem:

$$x^2 + y^2 = \ell^2$$

Taking derivatives: $2x\dot{x} + 2y\dot{y} = 0$

Hence: $\dot{y} = -\frac{x}{y}\dot{x}$

Now, when $x = 12\,\text{ft} \Rightarrow y^2 = \ell^2 - x^2$

$\Rightarrow \quad y^2 = 13^2 - 12^2 = 169 - 144 = 25$

$\Rightarrow \quad y = 5\,ft.$

So, $\quad \dot{y} = -\frac{x}{y}\dot{x} = -\frac{12\,\cancel{ft}}{5\,\cancel{ft}} \cdot 5\frac{ft}{sec} = -12\,ft/sec.$

$$\boxed{\dot{y} = -12\,ft/sec}$$

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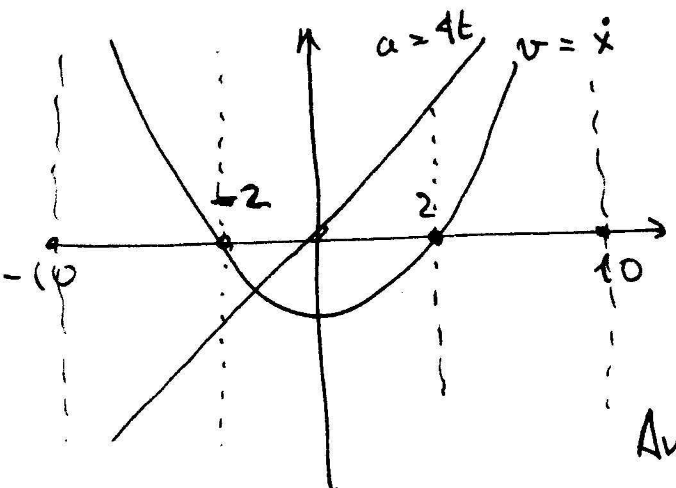
(3) Position: $\quad x(t) = \frac{2}{3}t^3 - 8t$

Velocity: $\quad \dot{x}(t) = 2t^2 - 8 = 2(t^2 - 4)$

Acceleration: $\quad \ddot{x}(t) = 4t.$



When $\dot{x}$ and $\ddot{x}$ have the same sign, the driver put gas (acelero). That is, when

$$t \in (-2, 0) \cup (2, 10]$$

And the driver brakes (freno) when $\dot{x}$ and $\ddot{x}$ have opposite signs, ie., when,

$$t \in [-10, -2) \cup (0, 2).$$

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(4) Since the derivatives of $y = \cos x$ repeats every four derivatives, i.e., $y = y^{(4)} = y^{(8)} = \cdots = y^{(40)} = \cos x$.

Now $\quad y^{(41)} = \frac{d}{dx}\left(y^{(40)}(x)\right) = \frac{d}{dx}\cos x = -\sin x$

$$\Rightarrow \quad \boxed{\frac{d^{41}}{dx^{41}}\cos x = 0}$$

(5) (a) Compute $\frac{dy}{dx}$ if $y(x)$ is implicitly given by:

$$x^4 + \sin y = x^3 y^2 .$$

Taking derivatives on both members of the equation:

$$\frac{d}{dx}\left(x^4 + \sin y\right) = \frac{d}{dx}\left(x^3 y^2\right)$$

By the product rule and the chain rule:

$$4x^3 + (\cos y)\,y' = 3x^2 \cdot y^2 + x^3\, 2y y'$$

$$\Rightarrow \quad \left(\cos y - 2x^3 y\right) y' = 3x^2 y^2 - 4x^3$$

$$\Rightarrow \quad \boxed{y' = \frac{3x^2y^2 - 4x^3}{\cos y - 2x^3 y}}$$

(b) The points (π^2, π) belongs to the graph of $y(x)$, since the equation $x^4 + \sin y = x^3 y^2$ holds at (π^2, π):

$$x^4 + \sin y = (\pi^2)^4 + \sin \pi = \pi^8 + 0 = \pi^8$$

$$x^3 y^2 = (\pi^2)^3 (\pi)^2 = \pi^6 \pi^2 = \pi^8$$

and they are equal.

Now, the slope of the tangent line is:

$$m = \left.\frac{d}{dx} y\right|_{\substack{x=\pi^2 \\ y=\pi}} = \frac{3(\pi^2)^2(\pi)^2 - 4(\pi^2)^3}{\cos(\pi) - 2(\pi^2)^3(\pi)} = \frac{3\pi^4\cdot\pi^2 - 4\pi^6}{-1-2\pi^6\cdot\pi}$$

$$= \frac{3\pi^6 - 4\pi^6}{-1-2\pi^7} = \frac{-\pi^6}{-1-2\pi^7} = \frac{\pi^6}{1+2\pi^7}$$

Hence, the slope of the normal line is: $m_\perp = \frac{-1}{m} = -\frac{(1+2\pi^7)}{\pi^6}$

Hence, the equation of the normal line is.

$$\boxed{y - \pi = -\left(\frac{1+2\pi^{7}}{\pi^{6}}\right)(x - \pi^{2})}$$

It cannot be simplified further.

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(6) The derivative of $g(x)$ is.

$$g'(x) = \sec^2\left(\frac{1}{x^2} + \cos\left[x^5 + (1+x^4)^{5/3}\right]\right) \cdot \frac{d}{dx}\left[\frac{1}{x^2} + \cos\left[x^5 + (1+x^4)^{5/3}\right]\right]$$

$$= \sec^2\left(\frac{1}{x^2} + \cos\left[x^5 + (1+x^4)^{5/3}\right]\right)\left[\frac{-2}{x^3} - \sin\left[x^5 + (1+x^4)^{5/3}\right] \cdot \frac{d}{dx}\left(x^5 + (1+x^4)^{5/3}\right)\right]$$

$$= -\sec^2\left(\frac{1}{x^2} + \cos\left[x^5 + (1+x^4)^{5/3}\right]\right)\left[\frac{2}{x^3} + \sin\left(x^5 + (1+x^4)^{5/3}\right)\left(5x^4 + \frac{5}{3}(1+x^4)^{2/3} 4x\right)\right]$$

$$\boxed{g' = -\sec^2\left(\frac{1}{x^2} + \cos\left[x^5 + (1+x^4)^{5/3}\right]\right)\left[\frac{2}{x^3} + \sin\left(x^5 + (1+x^4)^{5/3}\right)\left(5x^4 + \frac{20}{3}x(1+x^4)^{2/3}\right)\right]}$$
