Cálculo Diferencial UAM Azcapotzalco. Vie, Junio 3, 2016.

Examen # 1. Answer key

(1) (a) $\frac{df}{dx} = \frac{d}{dx}\left(\frac{3}{x^2} + x^{5/3}\right) = 3\left(\frac{-2}{x^3}\right) + \frac{5}{3}x^{\frac{5}{3}-1}$

$= -\frac{6}{x^3} + \frac{5}{3}x^{2/3} = -\frac{6}{x^3} + \frac{5}{3}\sqrt[3]{x^2}$.

(b) $\frac{dg}{dx} = \frac{d}{dx}\left(\frac{4-3x}{3x^2+3}\right) = \frac{(3x^2+3)(4-3x)' - (4-3x)(3x^2+3)'}{(3x^2+3)^2}$

$= \frac{(3x^2+3)(-3) - (4-3x)6x}{(3x^2+3)^2} = \frac{-9x^2-9-24x+18x^2}{(3x^2+3)^2}$

$$\boxed{\frac{dg}{dx} = \frac{9x^2-24x-9}{3^2(x^2+1)^2}}$$

(2) (i) Calculate $f(x+h) - f(x) = \frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}} = \frac{\sqrt{x} - \sqrt{x+h}}{\sqrt{x+h}\sqrt{x}} =$

$= \frac{\sqrt{x} - \sqrt{x+h}}{\sqrt{x+h}\sqrt{x}} \cdot \frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}} = \frac{(x)-(x+h)}{(\sqrt{x+h}\sqrt{x})(\sqrt{x}+\sqrt{x+h})} =$

$= \frac{-h}{(\sqrt{x+h}\sqrt{x})(\sqrt{x}+\sqrt{x+h})}$

(ii) Compute $\frac{f(x+h)-f(x)}{h} = \frac{1}{h}\frac{-h}{(\sqrt{x+h}\sqrt{x})(\sqrt{x}+\sqrt{x+h})} =$

$= \frac{-1}{(\sqrt{x+h}\sqrt{x})(\sqrt{x}+\sqrt{x+h})}$

(iii) Finally compute the limit:

$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{-1}{(\sqrt{x+h}\sqrt{x})(\sqrt{x}+\sqrt{x+h})} =$

$= 1 =$

$$= \frac{-1}{\sqrt{x}\sqrt{x}(\sqrt{x}+\sqrt{x})} = \frac{-1}{\sqrt{x}\sqrt{x}\,2\sqrt{x}} = \frac{-1}{2(\sqrt{x})^3}$$

Hence: $\boxed{\frac{df}{dx} = \frac{-1}{2\sqrt{x^3}}}$ or $\boxed{\frac{df}{dx} = \frac{-1}{2(\sqrt{x})^3}}$ or $\boxed{f' = -\frac{x^{-3/2}}{2}}$

---

(3) (a) Compute the composition of the functions:

$$f(x) = (F \circ G \circ \varphi)(x) = F(G(\varphi(x)))$$

$$= (G(\varphi(x)))^2 = (\sin(\varphi(x)))^2 = \left(\sin\left(\frac{1}{x}\right)\right)^2 \text{ ie } \sin^2\left(\frac{1}{x}\right)$$

(b) Compute $\frac{df}{dx}(x)$. We make use of the Chain rule:

$$\frac{df}{dx} = \frac{dF}{dv} \circ \frac{dv}{du} \circ \frac{du}{dx} = \frac{dF}{dv}\frac{dG}{du} \circ \frac{d\varphi}{dx}.$$

$$= \frac{d}{dv}v^2 \circ \frac{d}{du}\sin u \cdot \frac{d}{dx}\left(\frac{1}{x}\right) = (2v)\cos u\left(\frac{-1}{x^2}\right)$$

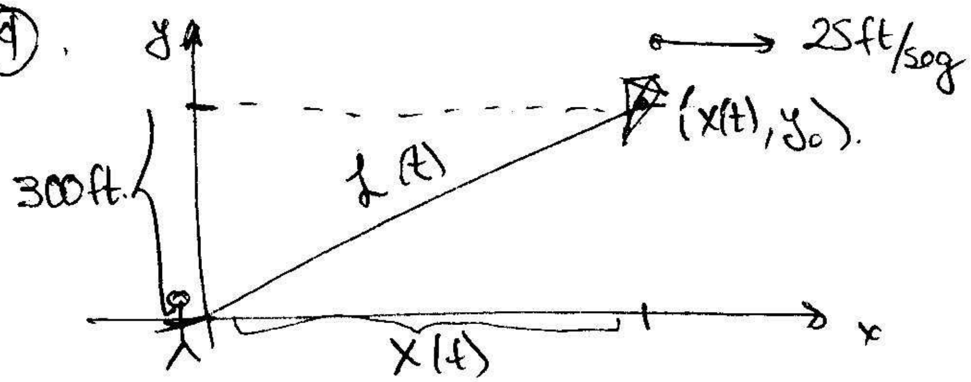
ie.

$$\frac{df}{dx} = 2(\sin u)\cos u\left(\frac{-1}{x^2}\right)$$

ie. $\boxed{\frac{df}{dx} = \frac{-2}{x^2}\sin\left(\frac{1}{x}\right)\cos\left(\frac{1}{x}\right)}$

---

(4).



$(x(t), y_0)$ are the coordinates of the comet.

Notice that.

$y_0 = 300$ ft $=$ const

$x(t)$ is function of time.

Also notice that the length of the string is $\ell(t)$, and it is function of time since the girl is realising the string.

$\ell(t)$ is also the distance (at time t) between the girl and the comet.

Now the equation that relates the dependent variables

$x(t)$

$\ell(t)$

is the Pythagorean theorem.

$$(*) \qquad \ell^2(t) = x^2(t) + y_0^2 \qquad (*)$$

We want to know $\frac{d\ell}{dt}$, so we compute derivatives:

$$\frac{d}{dt}\ell^2(t) = \frac{d}{dt}\left(x^2(t) + y_0^2\right) \Rightarrow 2\ell\frac{d\ell}{dt} = 2x\frac{dx}{dt} + 0$$

$$\Rightarrow \boxed{\ell\frac{d\ell}{dt} = x\frac{dx}{dt}}$$ This is the related rates equation.

Now, we want $\frac{d\ell}{dt} = \frac{x}{\ell}\frac{dx}{dt}$.

We know $\frac{dx}{dt} = 25\,ft/seg$, $\ell = 500\,ft$ (at same time t)

We compute x (at time t) using $(*)$ above:

$$x = \sqrt{\ell^2 - y_0^2} = \sqrt{(500)^2 - (300)^2} = 100\sqrt{5^2 - 3^2}$$
$$= 100\sqrt{25-9} = 100\sqrt{16} = 400\,ft.$$

(<u>No</u> calculator was needed!)

Hence: $\frac{d\ell}{dt} = \frac{x}{\ell}\frac{dx}{dt} = \frac{400(ft)\cdot 25}{500(ft)} ft/seg = \frac{4}{5}\cdot 25\ ft/seg$

$= 4\cdot 5\ ft/seg \Rightarrow \boxed{\frac{d\ell}{dt} = 20\ ft/seg}$

(No calculator was needed!)

---

(5) We have the implicit equation: $2xy + \pi \sin y = 2\pi$.

(a) At $(x,y) = (1, \frac{\pi}{2})$, we have:

$$2xy + \pi \sin y = 2(1)\cdot\frac{\pi}{2} + \pi \sin\left(\frac{\pi}{2}\right) = \pi + \pi\cdot 1 = \pi + \pi$$
$$= 2\pi$$ ✓ The equation holds

(b) We need implicit differenciation:

$$\frac{d}{dx}\left(2xy + \pi \sin y\right) = \frac{d}{dx}(2\pi)$$

i.e. $$2\frac{d}{dx}(xy) + \pi\frac{d}{dx}(\sin y) = 0.$$

By product rule and Chain Rule:

$$2\left(\frac{dx}{dx}\cdot y + x\frac{dy}{dx}\right) + \pi\frac{d}{dy}(\sin y)\frac{dy}{dx} = 0.$$

i.e. $$2y + 2x\frac{dy}{dx} + \pi\cos y\frac{dy}{dx} = 0.$$

i.e. $$(2x + \pi\cos y)\frac{dy}{dx} = -2y$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{-2y}{2x + \pi\cos y}}$$

(c) At $(1, \frac{\pi}{2})$, the slope of the tangent line is given by:

$$m = \left.\frac{dy}{dx}\right|_{(1, \frac{\pi}{2})} = \frac{-2(\pi/2)}{2\cdot 1 + \pi\cos\left(\frac{\pi}{2}\right)} = \frac{-\pi}{2+\pi\cdot 0}.$$

$$\Rightarrow \quad m = -\frac{\pi}{2}$$

The slope of the normal line is $m_{perp} = \frac{-1}{m} = \frac{2}{\pi}$.

Hence, the equation of the normal line in point-slope form is:

$$y - y_0 = m_{perp}(x - x_0)$$

i.e., for our case:

$$\boxed{y - \frac{\pi}{2} = \frac{2}{\pi}(x-1)}$$