UNIVERSIDAD AUTÓNOMA METROPOLITANA - AZCAPOTZALCO
CÁLCULO INTEGRAL

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EXAMEN #2 – A
FECHA: JUEVES 30A DE JUNIO DE 2016.

Nombre: Answer Key

Instrucciones.

(1) El examen consta de SEIS problemas Total: 100 puntos.
(2) Para recibir el total de puntos por problema, escriba de forma clara y concisa. Entregue su trabajo limpio y con sus ideas en orden, simplifique sus respuestas, muestre sus cuentas y **EXPLIQUE** su argumento.
(3) Apague y guarde su teléfono celular o tableta. Retiraré el examen y yo decidiré sobre su calificación a quienes sorprenda usádolos durante el mismo.

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(1) **(10 puntos).** Encuentre la forma de las fracciones parciales del siguiente cociente de polinomios. (No calcule los coeficientes. No integre.)

$$Q(x) = \frac{2x^7 + 5x + 1}{(x^2 + 4x + 8)^2 (x^2 - 6x + 9)^2 (x^4 - 16)}.$$

(2) ***(10 puntos).*** *Encuentre la antiderivada de la siguiente función:* $f(x) = \dfrac{2x^3 - 2x^2 + 1}{x^2 - x}$.

(3) **(20 puntos).** Determine la siguiente integral

$$\int_1^2 \frac{1}{x(\operatorname{Log} x)^2}\, dx.$$

(4) **(20 puntos).** Encuentre la antiderivada de $g(x) = \sec^3 x \tan^3 x$.

(5) **(20 puntos).** Encuentre la antiderivada de $h(x) = \dfrac{x^3}{\sqrt{4 + x^2}}$.

(6) **(20 puntos).** Calcule la siguiente integral.

$$\int_{-\pi}^{\pi} \sec(-\cos^3 x) \csc\left(1 + \frac{1 - \cos^2 x}{\tan^2 x}\right) \frac{1}{1 + \tan^2 x} \sin x \, dx.$$

~~Friday~~ Thursday, June 30 2016.

Exam #2-A. Integral Calculus

Universidad Autónoma Metropolitana.-Azcapotzalco.

Solution Key.

(1) Notice that:

$$x^2+4x+8=(x+2)^2+4 \quad \text{is irreducible}$$

$$x^2-6x+9=(x-3)^2$$

$$x^4-16=(x^2-4)(x^2+4)=(x-2)(x+2)(x^2+4)$$

and (x^2+4) is irreducible.

This way:

$$\oslash(x)=\frac{2x^7+5x+1}{((x+2)^2+4)^2(x-3)^4(x-2)(x+2)(x^2+4)}$$

and since degree(numerator) $<$ degree(denominator), we have the following partial fractions:

$$\boxed{\oslash(x)=\frac{Ax+B}{(x+2)^2+4}+\frac{Cx+D}{((x+2)^2+4)^2}+\frac{E}{x-3}+\frac{F}{(x-3)^2}+\frac{G}{(x-3)^3}+\frac{H}{(x-3)^4}+\frac{I}{x-2}+\frac{J}{x+2}+\frac{Kx+L}{x^2+4}.}$$

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(2) Since degree(numerator) $>$ degree(denominator), we must perform a polynomial division:

$$\begin{array}{r} 2x \\ x^2-x\ \overline{)\ 2x^3-2x^2+1} \\ \underline{2x^3-2x^2} \\ 0\ +1 \end{array} \quad \Rightarrow \quad f(x)=2x+\frac{1}{x^2-x}$$

Also observe: $f(x)=\dfrac{2x^3-2x^2+1}{x^2-x}=\dfrac{2x(x^2-x)+1}{x^2-x}$

$$=\frac{2x(x^2-x)}{x^2-x}+\frac{1}{x^2-x}=2x+\frac{1}{x^2-x} \Rightarrow f(x)=2x+\frac{1}{x^2-x}$$

$=f=$

Same result

The anti-derivative is

$$\int f(x)dx = \int 2x + \frac{1}{x^2-x}\,dx = x^2 + \int \frac{1}{x^2-x}\,dx.$$

We solve the last integral by partial fractions.

$$\frac{1}{x^2-x} = \frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} = \frac{A(x-1)+Bx}{x(x-1)}$$

$$\Rightarrow \quad 1 = A(x-1) + Bx$$

Take $\underline{x=1}$ $\quad 1 = 0 + B\cdot 1 \Rightarrow \boxed{B=1}$

$x=0 \quad 1 = A(-1)+0 \Rightarrow \boxed{A=-1}$

Hence: $$\int \frac{1}{x^2-x}\,dx = \int\left(\frac{-1}{x} + \frac{1}{x-1}\right)dx = -\mathrm{Log}|x| + \mathrm{Log}|x-1| + C$$

Therefore:

$$\boxed{\int \frac{2x^3-2x^2+1}{x^2-x}\,dx = x^2 + \mathrm{Log}\left|\frac{x-1}{x}\right| + C}$$

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3) The integral is improper since $\frac{1}{x(\mathrm{Log}\,x)^2}$ is <u>not</u> defined in $x=1$. Then, we should compute:

$$\lim_{\alpha\to 1^+}\int_\alpha^2 \frac{1}{x(\mathrm{Log}\,x)^2}\,dx.$$

The indefinite integral is:

$$\int \frac{1}{x(\mathrm{Log}\,x)^2}\,dx \underset{\substack{y=\mathrm{Log}\,x \\ dy=\frac{1}{x}dx}}{=} \int \frac{1}{y^2}\,dy = -y^{-1} + C = \frac{-1}{\mathrm{Log}|x|} + C$$

The definite integral becomes:

$$\int_\alpha^2 \frac{1}{x(\mathrm{Log}\,x)^2}\,dx = \frac{-1}{\mathrm{Log}\,x}\Big|_\alpha^2 = \frac{-1}{\mathrm{Log}\,2} + \frac{1}{\mathrm{Log}\,\alpha}.$$

And so:

$$\int_1^2 \frac{1}{x(\mathrm{Log}\,x)^2}\,dx = \lim_{\alpha\to1^+}\int_\alpha^2 \frac{1}{x(\mathrm{Log}\,x)^2}\,dx = \lim_{\alpha\to1^+}\left(\frac{-1}{\mathrm{Log}\,2} + \frac{1}{\mathrm{Log}\,\alpha}\right)$$

Since, $\lim_{\alpha\to1^+} \mathrm{Log}(\alpha) = 0^+$, then $\lim_{\alpha\to1^+} \frac{+1}{\mathrm{Log}\,\alpha} = +\infty.$

i.e.

$$\boxed{\int_1^2 \frac{1}{x(\mathrm{Log}\,x)^2}\,dx = +\infty}$$

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(4) We are required to compute

$$\int g(x)\,dx = \int \sec^3 x\,\tan^3 x\,dx = \int \sec^2 x\,\tan^2 x\,(\sec x\tan x)\,dx$$

$$= \int (\sec^2 x)(\sec^2 x - 1)(\sec x)'\,dx$$

This suggests: $y = \sec x$

$dy = (\sec x)'\,dx.$

$$= \int y^2(y^2-1)\,dy = \int (y^4 - y^2)\,dy = \frac{1}{5}y^5 - \frac{1}{3}y^3 + C$$

i.e.

$$\boxed{\int \sec^3 x\,\tan^3 x\,dx = \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C}$$

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(5) The antiderivative is the indefinite integral:

$\int \frac{x^3}{\sqrt{4+x^2}}\,dx$. Now, $4+x^2$ suggests $x = 2\tan\theta$, $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\Rightarrow dx = 2\sec^2\theta\,d\theta = \frac{2}{\cos^2\theta}\,d\theta$$

Also: $\sqrt{4+x^2} = \sqrt{4+4\tan^2\theta} = 2\sqrt{1+\tan^2\theta} = 2|\sec\theta| = \frac{2}{|\cos\theta|}$

In the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\cos\theta > 0 \Rightarrow |\cos\theta| = \cos\theta$.

Hence: $\sqrt{4+x^2} = \dfrac{2}{\cos\theta}$.

We then have:

$$\int \frac{x^3}{\sqrt{4+x^2}}\,dx = \int \frac{2^3\tan^3\theta}{\left(\frac{2}{\cos\theta}\right)}\left(\frac{2}{\cos^2\theta}\right)d\theta = 8\int \frac{\tan^3\theta}{\cos\theta}\,d\theta$$

$$= 8\int \frac{\sin^3\theta}{\cos^4\theta}\,d\theta = 8\int \frac{\sin^2\theta}{\cos^4\theta}\,\sin\theta\,d\theta$$

$$= 8\int \frac{1-\cos^2\theta}{\cos^4\theta}(-\cos\theta)'\,d\theta$$

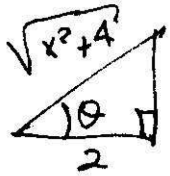
This suggests, $y = \cos\theta$

$dy = (\cos\theta)'\,d\theta$ i.e. $(-1)\,dy = (-\cos\theta)'\,d\theta$.

$$= 8\int \frac{1-y^2}{y^4}(-1)\,dy = 8\int\left(\frac{1}{y^2} - \frac{1}{y^4}\right)dy$$

$$= 8\left(-\frac{1}{y} + \frac{1}{3y^3}\right) = 8\left(-\frac{1}{\cos\theta} + \frac{1}{3\cos^3\theta}\right) + C$$

Now

$\tan\theta = \dfrac{x}{2} \Rightarrow$ 

$\Rightarrow \cos\theta = \dfrac{2}{\sqrt{x^2+4}} \Rightarrow \sec\theta = \dfrac{\sqrt{x^2+4}}{2}$

This way,

$$\int \frac{x^3}{\sqrt{4+x^2}}\,dx = 8\left(-\frac{\sqrt{x^2+4}}{2} + \frac{1}{3}\left(\frac{\sqrt{x^2+4}}{2}\right)^3\right) + C$$

i.e.
$$\boxed{\int \frac{x^3}{\sqrt{4+x^2}}\,dx = -4\sqrt{x^2+4} + \frac{1}{3}(x^2+4)^{3/2} + C}$$

(6) (i) The function $\cos x$ is <u>even</u>. Then $-\cos^3 x$ is <u>even</u> and, consequently, $\sec(-\cos^3 x)$ is <u>even</u>

(ii) $\tan x$ is odd, but $\tan^2 x$ is <u>even</u>. Also, $1-\cos^2 x$ is <u>even</u>. Then $1 + \frac{1-\cos^2 x}{\tan^2 x}$ is <u>even</u>

and also $\csc\left(1+\frac{1-\cos^2 x}{\tan^2 x}\right)$ is <u>even</u>

(iii) $\tan x$ is odd, $\tan^2 x$ is <u>even</u>, then $\frac{1}{1+\tan^2 x}$ is <u>even</u>

Therefore:

$$\sec(-\cos^3 x)\csc\left(1+\frac{1-\cos^2 x}{\tan^2 x}\right)\frac{1}{1+\tan^2 x} \quad \text{is } \underline{\text{even}}$$

Then,

$$\underbrace{\sec(-\cos^3 x)\csc\left(1+\frac{1-\cos^2 x}{\tan^2 x}\right)\frac{1}{1+\tan^2 x}}_{\text{Even function}}\ \underbrace{\sin x}_{\text{odd function}} \text{ is an odd}$$

is an <u>odd</u> function, since $\sin x$ is and <u>odd</u> function. <u>function</u>,
And if we integrated on $[-\pi,\pi]$ (a symmetric interval with respect $x=0$), we get:

$$\boxed{\int_{-\pi}^{\pi} \sec(-\cos^3 x)\csc\left(1+\frac{1-\cos^2 x}{\tan^2 x}\right)\frac{1}{1+\tan^2 x}\sin x\, dx = 0}$$

<u>Remark 1</u> $1 + \frac{1-\cos^2 x}{\tan^2 x} = 1 + \frac{\sin^2 x}{\tan^2 x} = 1+\cos^2 x$ is <u>even</u>.

<u>Remark 2</u>: $\csc(x)$ and $\sin(x)$ are <u>not</u> inverse functions!!!

$$\csc(\sin(x)) \neq x, \quad \sin(\csc(x)) \neq x.$$

Instead $\csc(\sin(x)) = \frac{1}{\sin(\sin x)}$; $\sin(\csc(x)) = \sin\left(\frac{1}{\sin x}\right)$.