UNIVERSIDAD AUTÓNOMA METROPOLITANA - AZCAPOTZALCO
CÁLCULO INTEGRAL

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EXAMEN #2 – B.
FECHA: JUEVES 30B DE JUNIO DE 2016.

Nombre: Answer Key

Instrucciones.

(1) El examen consta de SEIS problemas Total: 100 puntos.
(2) Para recibir el total de puntos por problema, escriba de forma clara y concisa. Entregue su trabajo limpio y con sus ideas en orden, simplifique sus respuestas, muestre sus cuentas y **EXPLIQUE** su argumento.
(3) Apague y guarde su teléfono celular o tableta. Retiraré el examen y yo decidiré sobre su calificación a quienes sorprenda usádolos durante el mismo.

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(1) **(10 puntos).** Encuentre la forma de las fracciones parciales del siguiente cociente de polinomios. (No calcule los coeficientes. No integre.)

$$Q(x) = \frac{2x^7 + 5x + 1}{(2x^2 + 2x + 2)^3(2x^2 + 2x + 1/2)(x^3 - 27)}.$$

(2) **(10 puntos).** *Encuentre la antiderivada de la siguiente función:* $f(x) = \dfrac{3x^4 - 3x^3 - 2}{x^2 - x}$.

(3) **(20 puntos).** Determine la siguiente integral

$$\int_2^\infty \frac{1}{x(\operatorname{Log} x)^2}\, dx.$$

(4) **(20 puntos).** Encuentre la antiderivada de $g(x) = \sec^4 x \tan^4 x$.

(5) **(20 puntos).** Encuentre la antiderivada de $h(x) = \dfrac{1}{x^2\sqrt{4 + x^2}}$.

(6) **(20 puntos).** Calcule la siguiente integral.

$$\int_{-\pi}^{\pi} e^{-\cos^3 x} \operatorname{Log}\left(1 + \frac{1 - \cos^2 x}{\tan^2 x}\right) \frac{1}{1 + \tan^2 x} \sin x\, dx.$$

Exam #2 - B INTEGRAL CALCULUS Thursday, June 30, 2016

Universidad Autónoma Metropolitana - Azcapotzalco

① Note that:

$2x^2+2x+1 = 2(x^2+x+1) = 2\left[\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right]$ is irreducible.

$2x^2+2x+\frac{1}{2} = 2\left(x^2+x+\frac{1}{4}\right) = 2\left(x+\frac{1}{2}\right)^2$ is its factorization.

$(x^3-27) = (x-3)(x^2+3x+9)$ is its factorization

with $x^2+3x+9 = \left(x+\frac{3}{2}\right)^2+\frac{27}{4}$ irreducible

Hence:

$$Q(x) = \frac{2x^7+5x+1}{2^3\left(\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right)^3 2\left(x+\frac{1}{2}\right)^2(x-3)\left[\left(x+\frac{3}{2}\right)^2+\frac{27}{4}\right]}$$

and degree(numerator) < degree(denominator) ($7 < 11$).

$$\boxed{Q(x) = \frac{Ax+B}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}} + \frac{Cx+D}{\left(\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right)^2} + \frac{Ex+F}{\left(\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right)^3} + \frac{G}{x+\frac{1}{2}} + \frac{H}{\left(x+\frac{1}{2}\right)^2} + \frac{I}{x-3} + \frac{Jx+K}{\left(x+\frac{3}{2}\right)^2+\frac{27}{9}}}$$

② $f(x) = \dfrac{3x^4-3x^3-2}{x^2-x}$. Notice that: $4 > 2$

degree (numerator) > degree (denominator)

Then, we must divide the polynomials:

$$\begin{array}{r|l} & 3x^2 \\ x^2-x & 3x^4-3x^3-2 \\ & 3x^4-3x^3 \\ \hline & 0\quad -2 \end{array} \Rightarrow f(x) = 3x^2 - \frac{2}{x^2-x}.$$

Similarly, $\dfrac{3x^4-3x^3-2}{x^2-x} = \dfrac{3(x^2)(x^2-x)-2}{x^2-x} =$

$= \dfrac{3x^2(x^2-x)}{(x^2-x)} - \dfrac{2}{x^2-x} = 3x^2 - \dfrac{2}{x^2-x}$ Same result

$=1=$

Hence: $\int f(x)dx = \int 3x^2 dx - 2\int \frac{1}{x^2-x} dx.$

$$= x^3 - 2\int \frac{1}{x(x-1)} dx$$

Partial fraction:

$$\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} = \frac{A(x-1)+Bx}{x(x-1)}$$

$$\Rightarrow \quad 1 = A(x-1)+Bx$$

Take $x=1$ | $1 = 0 + B$ $\Rightarrow$ $\boxed{B=1 \quad A=-1}$

$x=0$ | $1 = A(-1)+0$

$$\Rightarrow \int f(x)dx = x^3 - 2\left[\int\left(-\frac{1}{x}+\frac{1}{x-1}\right)dx\right]$$

$$= x^3 + 2\log|x| - 2\log|x-1| + C$$

$$\Rightarrow \boxed{\int \frac{3x^4-3x^3-2}{x^2-x}dx = x^3 + \log\left(\frac{x}{x-1}\right)^2 + C}$$

③ The integral $\int_2^\infty \frac{1}{x(\log x)^2}dx$ is improper.

Then:

$$\int_2^\infty \frac{1}{x(\log x)^2}dx = \lim_{M\to\infty}\int_2^M \frac{1}{x(\log x)^2}dx.$$

Compute the indefinite integral.

$$\int \frac{1}{x(\log x)^2}dx = \int \frac{1}{y^2}dy = -\frac{1}{y} + C. = \frac{-1}{\log x} + C$$

$y = \log x, \quad dy = \frac{1}{x}dx$

The definite integral is:

$$\int_2^M \frac{1}{x(\log x)^2}dx = \frac{-1}{\log x}\Big|_2^M = -\frac{1}{\log M} + \frac{1}{\log 2}.$$

Now, $\lim_{M\to\infty} \operatorname{Log} M = \infty \Rightarrow \lim_{M\to\infty} \frac{1}{\operatorname{Log} M} = 0$.

Hence,

$$\int_2^\infty \frac{1}{x(\operatorname{Log} x)^2}\,dx = \lim_{M\to\infty} \int_2^M \frac{1}{x(\operatorname{Log} x)^2}\,dx = \lim_{M\to\infty}\left(\frac{-1}{\operatorname{Log} M} + \frac{1}{\operatorname{Log} 2}\right)$$

$$\Rightarrow \boxed{\int_2^\infty \frac{1}{x(\operatorname{Log} x)^2}\,dx = \frac{1}{\operatorname{Log} 2}}$$

(4) $\int g(x)\,dx = \int \sec^4 x \tan^4 x\,dx = \int \sec^2 x \tan^4 x \sec^2 x\,dx$

$$= \int (1+\tan^2 x)(\tan^4 x)(\tan x)'\,dx$$

This suggests, $y = \tan x$, $dy = (\tan x)'dx$.

$$= \int (1+y^2)\,y^4\,dy = \int y^4 + y^6\,dy = \frac{y^5}{5} + \frac{y^7}{7} + C$$

ie

$$\boxed{\int \sec^4 x \tan^4 x\,dx = \frac{\tan^5 x}{5} + \frac{\tan^7 x}{7} + C}$$

(5) $\int \frac{1}{x^2\sqrt{4+x^2}}\,dx$. This suggests: $x = 2\tan\theta$, $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$dx = 2\sec^2\theta\,d\theta$.

$$\sqrt{4+x^2} = \sqrt{4+4\tan^2\theta} = 2\sqrt{1+\tan^2\theta} = 2|\sec\theta| = \frac{2}{|\cos\theta|}$$

$= \frac{2}{\cos\theta} = 2\sec\theta$, since $\cos\theta > 0$ in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and so $|\cos\theta| = \cos\theta$.

This way:

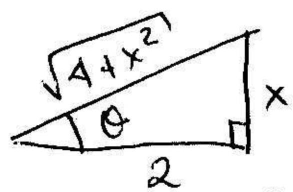
$$\int \frac{1}{x^2\sqrt{4+x^2}}\,dx = \int \frac{1}{4\tan^2\theta\, 2\sec\theta}\, 2\sec^2\theta\, d\theta = \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\, d\theta$$

$$= \frac{1}{4}\int \frac{\cos^2\theta}{\sin^2\theta}\,\frac{1}{\cos\theta}\, d\theta = \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\, d\theta$$

$$= \frac{1}{4}\int \frac{(\sin\theta)'}{\sin^2\theta}\, d\theta = \frac{1}{4}\int \frac{d}{d\theta}\left(\frac{-1}{\sin\theta}\right) d\theta$$

$$= -\frac{1}{4}\,\frac{1}{\sin\theta} + C.$$

Now,

$\tan\theta = \frac{x}{2} \Rightarrow$ 

$\Longrightarrow \sin\theta = \frac{x}{\sqrt{x^2+4}}$

Hence:

$$\boxed{\int \frac{1}{x^2\sqrt{4+x^2}}\,dx = -\frac{1}{4}\,\frac{\sqrt{x^2+4}}{x} + C}$$

(6) (i) $\cos x$ is an even function. Then, $-\cos^3 x$ is even and $\exp(-\cos^3 x)$ is also even.

(ii) $\tan x$ is odd, but $\tan^2 x$ is even. Then $\frac{1-\cos^2 x}{\tan^2 x}$ is even and so, $\operatorname{Log}\left(1+\frac{1-\cos^2 x}{\tan^2 x}\right)$ is even.

(iii) $\tan x$ is odd, but $\tan^2 x$ and $\frac{1}{1+\tan^2 x}$ are even.

Therefore,

$$e^{-\cos^3 x}\operatorname{Log}\left(1+\frac{1-\cos^2 x}{\tan^2 x}\right)\frac{1}{1+\tan^2 x} \text{ is even.}$$

$A =$

and consequently

$$\underbrace{e^{-\cos^3 x} \operatorname{Log}\left(1+\frac{1-\cos^2 x}{\tan^2 x}\right)\frac{1}{1+\tan^2 x}}_{\text{even function}} \underbrace{\sin x}_{\text{odd function}}$$ is an odd function

is an <u>odd function</u> since, $\sin(x)$ is an odd function.

If we integrate this function in a symmetric interval $[-\pi, \pi]$, we get:

$$\boxed{\int_{-\pi}^{\pi} e^{-\cos^3 x} \operatorname{Log}\left(1+\frac{1-\cos^2 x}{\tan^2 x}\right)\frac{1}{1+\tan^2 x} \sin x \, dx = 0}$$

<u>Remark</u> $1+\dfrac{1-\cos^2 x}{\tan^2 x} = 1+\dfrac{\sin^2 x}{\tan^2 x} = 1+\cos^2 x$ is an <u>even function</u>, and is well defined in $[-\pi,\pi]$