UAM - Azcapotzalco. Differential Calculus Thursday September 1, 2011

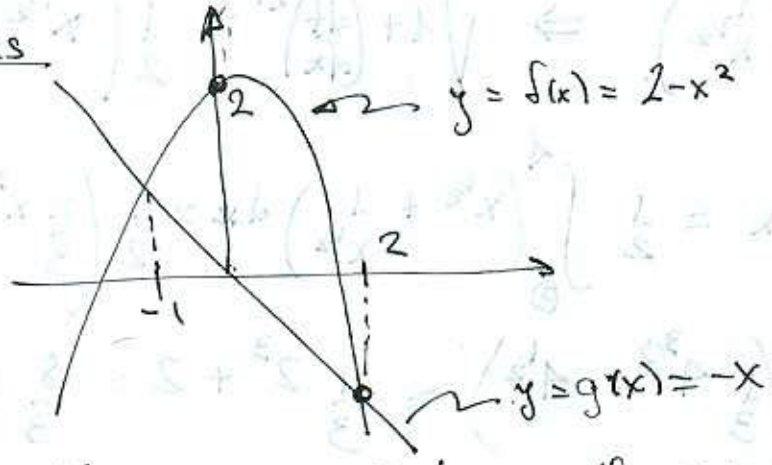
Exam # 3

(1) Intersection points: $-x = 2 - x^2$

$\Rightarrow x^2 - x - 2 = 0 \Rightarrow (x-2)(x+1) = 0$

$\Rightarrow \boxed{\begin{matrix} x = -1 \\ x = 2 \end{matrix}} \Rightarrow \begin{matrix} y = 1 \\ y = -2 \end{matrix}$ Intersection points $(-1, 1)$ and $(2, -2)$

Graphs



Notice that:

$f(x) \geq g(x)$ on $[-1, 2]$

then:

$f(x) - g(x) \geq 0$ in $[-1, 2]$

Hence, the area in between the curves is:

$$V = \int_{-1}^{2} (f(x) - g(x))\,dx = \int_{-1}^{2} (2 - x^2) - (-x)\,dx = \int_{-1}^{2} (2 - x^2 + x)\,dx.$$

$$= \left(2x - \frac{1}{3}x^3 + \frac{x^2}{2}\right)\Big|_{-1}^{2} = \left(4 - \frac{1}{3}8 + \frac{4}{2}\right) - \left(-2 + \frac{1}{3} + \frac{1}{2}\right)$$

$$= \left(\frac{24 - (16+12)}{3 \cdot 2}\right) - \left(\frac{-12 + 2 + 3}{6}\right) = \frac{20}{6} - \frac{-7}{6} = \frac{27}{6} = \frac{9}{2}.$$

$$\boxed{\text{Area} = \frac{27}{6} = \frac{9}{2}}$$

(2) We should compute

$$L = \int_0^1 \sqrt{1 + \left(\frac{df}{dx}\right)^2}\,dx$$

$= 1 =$

We require: $\frac{df}{dx} = \frac{1}{2}x^{1/2} - \frac{1}{2}x^{-1/2} = \frac{1}{2}\left(x^{1/2} - \frac{1}{x^{1/2}}\right)$

Now:

$$\left(\frac{df}{dx}\right)^2 = \frac{1}{4}\left(x^{1/2} - \frac{1}{x^{1/2}}\right)^2 = \frac{1}{4}\left(x - 2 + \frac{1}{x}\right).$$

and

$$1 + \left(\frac{df}{dx}\right)^2 = 1 + \frac{1}{4}\left(x - 2 + \frac{1}{x}\right) = \frac{1}{4}\left(4 + x - 2 + \frac{1}{x}\right) = \frac{1}{4}\left(x + 2 + \frac{1}{x}\right)$$

$$= \frac{1}{4}\left(x^{1/2} + \frac{1}{x^{1/2}}\right)^2 \Rightarrow \sqrt{1 + \left(\frac{df}{dx}\right)^2} = \frac{1}{2}\left(x^{1/2} + \frac{1}{x^{1/2}}\right)$$

Hence:

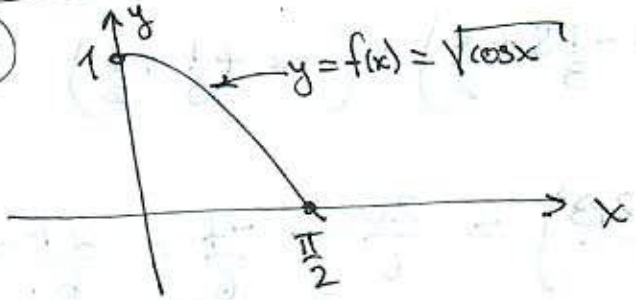
$$L = \int_0^4 \sqrt{1 + \left(\frac{df}{dx}\right)^2}\, dx = \frac{1}{2}\int_0^4 \left(x^{1/2} + \frac{1}{x^{1/2}}\right) dx = \frac{1}{2}\left(\frac{2}{3}x^{3/2} + 2x^{1/2}\right)\Big|_0^4$$

$$= \left(\frac{1}{3}x^{3/2} + x^{1/2}\right)\Big|_0^4 = \left(\frac{1}{3}4^{3/2} + 4^{1/2}\right) = \frac{1}{3}2^3 + 2 = \frac{8}{3} + 2 = \frac{14}{3}$$

So, the length is: $\boxed{L = \frac{14}{3}}$

---

(3)



Since we consider a solid of revolution, the volume is

$$V = \pi\int_0^{\pi/2} f^2(x)\, dx = \pi\int_0^{\pi/2} (\sqrt{\cos x})^2\, dx$$

$$V = \pi\int_0^{\pi/2} \cos x\, dx = \pi(\sin x)\Big|_0^{\pi/2} = \pi\left(\sin\left(\frac{\pi}{2}\right) - \sin 0\right) =$$

$$= \pi(1 - 0).$$ The volume is

$$\boxed{V = \pi}$$

---

(4) Since $\frac{1}{10}$ m is the natural length, Hooke's law is $F = -k\left(x - \frac{1}{10}\right)$.

(a) When $x = \frac{14}{100}$ cm, $F = 800$ N. So:

$$800 = k\left|\frac{14}{100} - \frac{10}{100}\right| = k\,\frac{4}{100} \Rightarrow k = \frac{800 \cdot 100}{4} = 200 \cdot 100$$

$$\Rightarrow \boxed{k = 20{,}000 \text{ N/m}}$$

(b) We require to compute:

$$W = -k\int_{14/100}^{18/100}\left(x - \frac{10}{100}\right)dx = -k\int_{4/100}^{8/100} u\,du = -\frac{k}{2}u^2\Big|_{\frac{4}{100}}^{8/100}$$

$$u = x - \frac{10}{100}$$

$$= -\frac{k}{2}\,\frac{8^2 - 4^2}{(100)^2} = -\frac{k}{2}\,\frac{64 - 16}{10^4} = -k\,\frac{32 - 8}{10^4} = -k\,\frac{24}{10^4}$$

$$\Rightarrow W = -(2\times 10^4)\left(\frac{24}{10^4}\right) \Rightarrow \boxed{W = -48 \text{ Joules}}$$