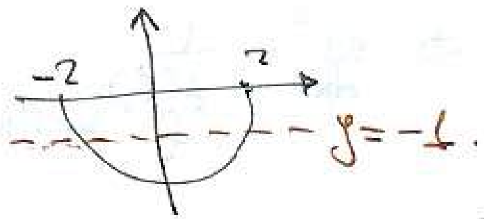
UAM Azcapotzalco Differential Calculus Friday, September 2, 2016

Exam #3.

① (a). $Dom(f) = [-2, 2]$, since $4 - x^2 \geq 0 \Rightarrow 4 \geq x^2$

$\Rightarrow \sqrt{x^2} \leq \sqrt{4} \Rightarrow |x| \leq 2 \Rightarrow -2 \leq x \leq 2$.

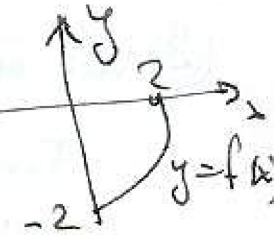
$Rang(f) = [2, 0]$, since $y = -\sqrt{4-x^2}$ is a semicircle



(b) The function is not injective since the horizontal line $y=-1$ crosses the graph twice.

(b) Restrict the domain to $Dom(f) = [0, 2]$ and keep $Rang(f)$ the same. (Remark. We can also take the restrict the $Dom(f) = [-2, 0]$)

The function is now injective and has an inverse function with $Dom(f^{-1}) = [-2, 0]$, $Ran(f^{-1}) = [0, 2]$.



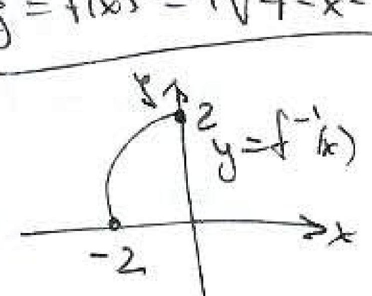
Computing f^{-1}:

f^{-1} $y = -\sqrt{4-x^2} \Rightarrow x = -\sqrt{4-y^2}$, by exchanging $x \leftrightarrow y$

$\Rightarrow x^2 = 4 - y^2 \Rightarrow y^2 = 4 - x^2 \Rightarrow y = \pm\sqrt{4-x^2}$. Choose "+",

since $y \in Rang(f) = [0, 2]$, $\Rightarrow y \geq 0 \Rightarrow$ $\boxed{y = f^{-1}(x) = +\sqrt{4-x^2}}$

(c) We are required to compute:



$$\frac{df^{-1}}{dx}(x) = \left.\frac{1}{\frac{df(y)}{dy}}\right|_{y=f^{-1}(x)}$$

Compute first $\frac{df}{dy}(y) = \frac{d}{dy}\left(-\sqrt{4-y^2}\right) = (-1)\frac{(-2y)}{2\sqrt{4-y^2}}$, by chain rule $= 1 =$

i.e. $\frac{df(y)}{dy} = \frac{y}{\sqrt{4-y^2}}$. Now: $\left.\frac{df(y)}{dy}\right|_{y=f^{-1}(x)} = \frac{f^{-1}(x)}{\sqrt{4-(f^{-1}(x))^2}} =$

$$= \frac{\sqrt{4-x^2}}{\sqrt{4-(\sqrt{4-x^2})^2}} = \frac{\sqrt{4-x^2}}{\sqrt{4-(4-x^2)}} = \frac{\sqrt{4-x^2}}{\sqrt{4-4+x^2}} = \frac{\sqrt{4-x^2}}{\sqrt{x^2}}$$

$= \frac{\sqrt{4-x^2}}{|x|}$. Since $x \in \mathrm{Dom}(f^{-1}) = [-2,0] \Rightarrow x<0$
$\Rightarrow |x| = -x$;

$\Rightarrow \left.\frac{df(y)}{dy}\right|_{y=f^{-1}(x)} = \frac{\sqrt{4-x^2}}{-x} \Rightarrow \frac{df^{-1}}{dx} = \frac{1}{\left.\frac{df(y)}{dy}\right|_{y=f^{-1}(x)}} = \frac{1}{\left(\frac{\sqrt{4-x^2}}{-x}\right)}$

$$\Rightarrow \boxed{\frac{df^{-1}}{dx}(x) = \frac{-x}{\sqrt{4-x^2}}}$$

This coincides with $\frac{df^{-1}}{dx} = \frac{d}{dx}\sqrt{4-x^2} = \frac{(-2x)}{2\sqrt{4-x^2}} = \frac{-x}{\sqrt{4-x^2}}$ ✓

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2. We are required to compute the derivatives of F at $a = -\frac{\pi}{2}$

$F(x) = \cos x$ $\qquad F(-\frac{\pi}{2}) = \cos(-\frac{\pi}{2}) = 0$

$F'(x) = -\sin x$ $\qquad F'(-\frac{\pi}{2}) = -\sin(-\frac{\pi}{2}) = 1$

$F''(x) = -\cos x$ $\qquad F''(-\pi/2) = -\cos(-\frac{\pi}{2}) = 0$

$F'''(x) = \sin x$ $\qquad F'''(-\pi/2) = \sin(-\frac{\pi}{2}) = -1$

The Taylor polynomial is:

$$\cos x = F(x) \approx 0 + 1\cdot\left(x+\frac{\pi}{2}\right) + 0 - \frac{1}{3!}\left(x+\frac{\pi}{2}\right)^3.$$

or

$$\cos x = f(x) \approx \left(x+\frac{\pi}{2}\right) - \frac{1}{3!}\left(x+\frac{\pi}{2}\right)^3.$$

Since $x = -92^\circ = -90^\circ - 2^\circ \Rightarrow x + 90^\circ = -2^\circ$.

Now $2^\circ = \frac{2\pi}{360}\cdot 2 = \frac{\pi}{90}$

$\Rightarrow x + 90^\circ = -2^\circ$ is equivalent to $x + \frac{\pi}{2} = -\frac{\pi}{90}$.

Hence:

$$\cos(-92)^\circ \approx \left(-\frac{\pi}{90}\right) - \frac{1}{3!}\left(-\frac{\pi}{90}\right)^3 =$$

$$= -\frac{\pi}{90} + \frac{1}{3!}\left(\frac{\pi}{90}\right)^3 \approx -0.034899\,49627064$$

i.e.

$$\cos(-92^\circ) \approx$$

Using directly the calculator: $\cos(-92^\circ) \approx -0.034899\,496702501$

This is up to 10^{-10} an estimation for the error

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③ (a) $\ln\sqrt{62.5} = \ln\sqrt{\frac{125}{2}} = \ln\left(\frac{5^3}{2}\right)^{1/2} = \frac{1}{2}\ln\left(\frac{5^3}{2}\right) =$

$= \frac{1}{2}\left(\ln(5^3) - \ln 2\right) = \frac{1}{2}\left(3\ln(5) - \ln(2)\right).$

(b) $G(x) = \cos\left((x^2)^{x^3}\right)$. We require $G'(x)$. By chain rule

$\frac{dG}{dx} = -\sin\left((x^2)^{x^3}\right) \cdot \frac{d}{dx}(x^2)^{x^3}$. We need to compute this last derivative. Define $g(x) = (x^2)^{x^3} = x^{2x^3}$. Then:

$\ln g = \ln(x^{2x^3}) = 2x^3 \ln x$. We can compute its derivative

$\frac{d}{dx}(\ln g) = \frac{d}{dx}(2x^3 \ln x) \Rightarrow \frac{g'}{g} = (2x^3)' \ln x + 2x^3 (\ln x)' \Rightarrow$

$\frac{g'}{g} = 6x^2 \ln x + 2x^3 \cdot \frac{1}{x} = 2x^2\, 3\ln x + 2x^2 = 2x^2(3\ln x + 1)$

$\Rightarrow g'(x) = 2x^2 g(x)(3\ln x + 1) \Rightarrow g'(x) = 2x^2 x^{2x^3}(3\ln x + 1)$

Hence

$$\boxed{\frac{dG}{dx} = -\sin(x^{2x^3})\, 2x^2 x^{2x^3}(3\ln x + 1)}$$

④ $\frac{dy}{dx} = \frac{d}{dx}\left(e^{(1+\operatorname{Arctan} x)} - \ln(\operatorname{Arccos} x)\right)$

$$= e^{(1+\operatorname{Arctan} x)} \frac{d}{dx}(1+\operatorname{Arctan} x) - \frac{1}{\operatorname{Arccos} x}\frac{d}{dx}\operatorname{Arccos} x,$$

by the chain rule,

$$= e^{(1+\operatorname{Arctan} x)}\left(0 + \frac{1}{1+x^2}\right) - \frac{1}{\operatorname{Arccos} x}\left(\frac{-1}{\sqrt{1-x^2}}\right).$$

i.e.

$$\frac{d}{dx}y = \frac{e^{(1+\operatorname{Arctan} x)}}{1+x^2} + \frac{1}{(\operatorname{Arccos} x)\sqrt{1-x^2}}.$$

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⑤ Direct evaluation leads to undetermined expression:

$$\left.\frac{t-1}{\sqrt{t}-1}\right|_{t=1} = \text{"}\frac{0}{0}\text{"}$$. Then, use L'Hôpital

$$\lim_{t\to 1}\frac{t-1}{\sqrt{t}-1} \underset{\text{L'Hôpital}}{=} \lim_{t\to 1}\frac{(t-1)'}{(\sqrt{t}-1)'} = \lim_{t\to 1}\frac{(1)}{\left(\frac{1}{2\sqrt{t}}\right)} =$$

$$= \lim_{t\to 1} 2\sqrt{t} = 2. \Rightarrow \boxed{\lim_{t\to 1}\frac{t-1}{\sqrt{t}-1} = 2}$$

By rationalization,

$$\lim_{t\to 1}\frac{t-1}{\sqrt{t}-1} = \lim_{t\to 1}\frac{t-1}{\sqrt{t}-1}\cdot\frac{(\sqrt{t}+1)}{(\sqrt{t}+1)} = \lim_{t\to 1}\frac{(t-1)(\sqrt{t}+1)}{(\sqrt{t})^2-1}$$

$$= \lim_{t\to 1}\frac{\cancel{(t-1)}(\sqrt{t}+1)}{\cancel{(t-1)}} = \lim_{t\to 1}(\sqrt{t}+1) = 2.$$ Same result.