EXAMEN GLOBAL DE CÁLCULO INTEGRAL.

Trimestre 16P Matutino

Parte PRIMERA.

$\boxed{1.}$ Calcular $G'(0)$ para la función $G(x) = (10x-7)\int_0^{\tan 9x} e^{-t^3}\,dt$.

Resp: Por la regla del producto:

$$\frac{dG}{dx} = \frac{d}{dx}(10x-7)\int_0^{\tan 9x} e^{-t^3}\,dt + (10x-7)\frac{d}{dx}\int_0^{\tan 9x} e^{-t^3}\,dt$$

$$= 10\int_0^{\tan 9x} e^{-t^3}\,dt + (10x-7)\, e^{-t^3}\Big|_{\tan 9x} \cdot \frac{d}{dx}\tan(9x),$$

por el Teorema Fundamental del Cálculo y la Regla de la Cadena

$$= 10\int_0^{\tan 9x} e^{-t^3}\,dt + (10x-7)\, e^{-\tan^3 9x} \cdot \frac{9}{1+(9x)^2}$$

Evaluando a $x=0$, $\tan(9x) = \tan 0 = 0$:

$$\frac{dG}{dx}(0) = 10\int_0^{0} e^{-t^3}\,dt + (-7)\,e^{-0}\cdot\frac{9}{1}$$

i.e. $\boxed{\frac{dG}{dx}(0) = -63}$

2 Calcular las siguientes integrales.

(a) $$\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = \int_a^b \frac{e^y}{y}\,2y\,dy = \int_a^b 2e^y\,dy = 2e^y\Big|_a^b$$

$y=\sqrt{x}$

$dy = \frac{1}{2\sqrt{x}}\,dx$

i.e. $dx = 2y\,dy$

$$= 2e^{\sqrt{x}}\Big|_1^4 = 2\left(e^{\sqrt{4}} - e^{\sqrt{1}}\right)$$

$$= 2\left(e^2 - e^1\right)$$

i.e. $$\boxed{\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2e(e-1)}$$

(b) $$\int_1^5 \left|\frac{3x-10}{x^2}\right| dx = \int_1^{10/3} -\frac{(3x-10)}{x^2}\,dx + \int_{10/3}^5 \left(\frac{3x-10}{x^2}\right)dx,$$

$$= -\int_1^{10/3}\left(\frac{3}{x} - \frac{10}{x^2}\right)dx + \int_{10/3}^5 \left(\frac{3}{x} - \frac{10}{x^2}\right)dx$$

$$= -\left[3\ln|x| + \frac{10}{x}\right]_1^{10/3} + \left[3\ln|x| + \frac{10}{x}\right]_{10/3}^5.$$

$$= \left[3\ln|x| + \frac{10}{x}\right]_{10/3}^1 + \left[3\ln|x| + \frac{10}{x}\right]_{10/3}^5$$

$$= \left(3\ln|1| + \frac{10}{1}\right) + \left(3\ln|5| + \frac{10}{5}\right) - 2\left(3\ln\left|\frac{10}{3}\right| + 10\cdot\frac{3}{10}\right).$$

$$= (0+10) + \left(3\ln|5| + 2\right) - 2\left(3\ln 2 + 3\ln 5 - 3\ln 3 + 3\right)$$

$$\boxed{\int_1^5 \frac{|3x-10|}{x^2}\,dx = 6 - 6\ln 2 + 6\ln 3 - 3\ln 5}$$

Calcular las siguientes integrales.

(a) $$\int \frac{7x + \operatorname{Arctan}^3 x}{1+x^2}\,dx = 7\int \frac{x}{1+x^2}\,dx + \int \frac{\operatorname{Arctan}^3 x}{1+x^2}\,dx$$

$u = 1+x^2$, $\quad v = \operatorname{Arctan} x$

$du = 2x\,dx$ $\quad dv = \frac{1}{1+x^2}\,dx$

$$= \frac{7}{2}\int \frac{1}{u}\,du + \int v^3\,dv = \frac{7}{2}\ln|u| + \frac{1}{4}v^4 + C$$

$$\boxed{= \frac{7}{2}\ln|1+x^2| + \frac{1}{4}\operatorname{Arctan}^4 x + C}$$

(b) $$\int_1^\infty \frac{\ln^2 x}{x^2}\,dx$$

Compute $$\int_1^M \frac{\ln^2 x}{x^2}\,dx = \int_0^{\ln M} \frac{1}{e^u}u^2\,du = \int_0^{\ln M} e^{-u}u^2\,du$$

$u = \ln x \quad du = \frac{1}{x}dx$

$$= -e^{-u}u^2\Big|_0^{\ln M} + 2\int_0^{\ln M} e^{-u}u\,du = \left[e^{-u}u^2 - 2e^{-u}u\right]_0^{\ln M} + 2\int_0^{\ln M} e^{-u}\,du$$

$$= \left(e^{-u}u^2 - 2e^{-u}u - 2e^{-u}\right)\Big|_0^{\ln M}$$

$$= e^{-u}(u^2 - 2u - 2)\Big|_0^{\ln M} = \frac{1}{M}(\ln^2 M - 2\ln M - 2) + 2.$$

Now $$\lim_{M\to\infty} \frac{\ln^2 M}{M} = \lim_{M\to 0} \frac{2\ln M \cdot \frac{1}{M}}{1} = \lim_{M\to\infty} \frac{2\ln M}{M}$$

L'Hôpital $$= \lim_{M\to\infty} \frac{2\frac{1}{M}}{1} = 0.$$

Hence:

$$\int_1^\infty \frac{\ln^2 x}{x^2}\,dx = \lim_{M\to\infty} \int_1^M \frac{\ln^2 x}{x^2}\,dx = \lim_{M\to\infty} \left[-\frac{1}{M}\left(\ln^2 M - 2\ln M - 2\right) + 2\right]$$

$$= 0 - 2\cdot 0 - 0 + 2.$$

$$\Rightarrow \boxed{\int_1^\infty \frac{\ln^2 x}{x^2}\,dx = 2}$$