Examen. Global de Cálculo Integral

Trimestre 16-P Vespertino

Parte Primera

① Calcular $F'(1)$, donde $F(x) = x\int_1^{x^2} \frac{t^6}{1+t^4}\,dt$.

Por la Regla del Producto.

$$\frac{dF}{dx}(x) = \frac{dx}{dx}\int_1^{x^2}\frac{t^6}{1+t^4}\,dt + x\frac{d}{dx}\int_1^{x^2}\frac{t^6}{1+t^4}\,dt,$$

$$= \int_1^{x^2}\frac{t^6}{1+t^4}\,dt + x\left(\frac{t^6}{1+t^4}\right)\Bigg|_{t=x^2}\cdot\frac{dx^2}{dx},$$ por el Teorema Fundamental del Cálculo y Regla de la Cadena

$$= \int_1^{x^2}\frac{t^6}{1+t^4}\,dt + x\cdot\frac{x^{12}}{1+x^8}\cdot 2x.$$

i.e

$$\frac{dF}{dx} = \int_1^{x^2}\frac{t^6}{1+t^4}\,dt + 2\frac{x^{14}}{1+x^8}.$$

$$\frac{dF(1)}{dx} = \int_1^1\frac{t^6}{1+t^4}\,dt + \frac{2}{1+1} \Rightarrow F'(1) = 0+1$$

$$\boxed{F'(1) = 1}.$$

$\boxed{2}$ Calcular las siguientes integrales.

(a) $$\int_1^e \frac{2+\ln u}{u}\,du = 2\int_1^e \frac{1}{u}\,du + \int_1^e \frac{\ln u}{u}\,du$$

$$= 2\ln|u|\Big|_1^e + \int_0^1 \frac{v}{e^v}\,e^v\,dv$$

$v = \ln u$

$e^v = u$

$e^v\,dv = du$

$$= 2\ln|u|\Big|_1^e + \int_0^1 v\,dv = 2\ln|u|\Big|_1^e + \frac{1}{2}v^2\Big|_0^1$$

$$= 2\Big(\ln(e) - \ln(1)\Big) + \frac{1}{2}(1^2 - 0^2)$$

i.e. $$\boxed{\int_1^e \frac{2+\ln u}{u}\,du = 2 + \frac{1}{2} = \frac{5}{2}}$$

(b) $$\int \frac{2x - e^{\operatorname{Arcsin}(2x)}}{\sqrt{1-4x^2}}\,dx = \int \frac{2x}{\sqrt{1-4x^2}}\,dx - \int \frac{e^{\operatorname{Arcsin}(2x)}}{\sqrt{1-4x^2}}\,dx$$

$u = 1-4x^2$	$v = \operatorname{Arcsin}(2x)$
$du = -8x\,dx$	$dv = \frac{1}{\sqrt{1-4x^2}}\,2\,dx$
$-\frac{1}{4}du = 2x\,dx$	$\frac{1}{2}dv = \frac{dx}{\sqrt{1-4x^2}}$

$$= \int \frac{-\frac{1}{4}du}{\sqrt{u}} - \int e^v\,\frac{1}{2}dv = \left(-\frac{1}{4}\right)2\sqrt{u} - \frac{1}{2}e^v + C$$

$$\boxed{= -\frac{1}{2}\sqrt{1-4x^2} - \frac{1}{2}e^{\operatorname{Arcsin}(2x)} + C}$$

$\boxed{3}$ Calcular las siguientes integrales:

(a) $$\int_{-\pi/4}^{\pi/2} \frac{\sin y \cos y}{9+\cos^4 y}\,dy = \int_{\pi/4}^{\pi/2} \frac{\sin y \cos y}{9+\cos^4 y}\,dy$$

since the integrand is an odd function

$$= \frac{1}{2}\int_{\pi/4}^{\pi/2} \frac{\sin(2y)}{9+(\cos^2 y)^2}\,dy = \frac{1}{2}\int_{\pi/4}^{\pi/2} \frac{\sin(2y)}{9+\left(\frac{1+\cos 2y}{2}\right)^2}\,dy$$

$x = \cos 2y$, $y \in [0, \frac{\pi}{2}]$.

$dx = -2\sin(2y)\,dy$

$-\frac{1}{2}dx = \sin(2y)\,dy$

$$= \frac{1}{2}\left(-\frac{1}{2}\right)\int_0^{-1} \frac{dx}{9+\left(\frac{1+x}{2}\right)^2}$$

Now let $3u = \frac{1+x}{2} \Rightarrow 6\,du = dx$

Así:

$$= -\frac{1}{4}6\int_{1/6}^{0} \frac{du}{9+9u^2} = \frac{3}{2}\left(\frac{1}{9}\right)\int_0^{1/6} \frac{du}{1+u^2} = \frac{1}{6}\operatorname{Arctan} u \Big|_0^{1/6}$$

$$\Rightarrow \boxed{\int_{-\pi/4}^{\pi/2} \frac{\sin y \cos y}{9+\cos^4 y}\,dy = \frac{1}{6}\operatorname{Arctan}\left(\frac{1}{6}\right)}$$

$\boxed{3}$(b)

$$\int z^{-3}\cos(z^{-1})\,dz \underset{\uparrow}{=} \int x^3\cos x\left(-\frac{1}{x^2}\,dx\right) = -\int x\cos x\,dx.$$

$$\frac{1}{z} = x.$$

$$-\frac{1}{z^2}\,dz = dx.$$

$$\Rightarrow\ dz = -z^2\,dx$$

$$= -\frac{1}{x^2}\,dx$$

$$= -x\sin x + \int \sin x\,dx = -x\sin x + \cos x + C$$

$$\boxed{\int z^{-3}\cos(z^{-1})\,dz = -\frac{1}{z}\sin\left(\frac{1}{z}\right) + \cos\left(\frac{1}{z}\right) + C.}$$