EXAM #2. Differential Calculus. Thursday, November 24, 2016

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(1) We have the function

$$f(x) = x^2 + x.$$

Hence:

$$f'(x) = 2x + 1$$

Now: $[a,b] = [2,5]$. Hence:

$$\frac{f(b)-f(a)}{b-a} = \frac{(5^2+5)-(2^2+2)}{5-2} = \frac{30-6}{3} = \frac{24}{3} = 8.$$

and $f'(c) = 2c+1$.

The Mean Value Theorem states that:

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$\Rightarrow \quad 2c+1 = 8 \quad \Rightarrow \quad \boxed{c = \frac{7}{2}}$$

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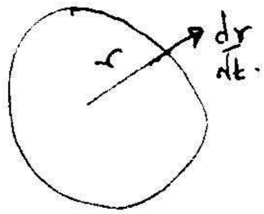
(2) (a) The disc has radius r and area $A = \pi r^2$.

Since r changes in time, A does so:



$$A(t) = \pi r^2(t)$$

$$\Rightarrow \quad \frac{dA}{dt} = \pi 2r \cdot \frac{dr}{dt} \quad (\text{by} \quad \text{rule.}$$

Since $v_0 = \frac{dr}{dt}$, then. $\boxed{\frac{dA}{dt} = 2\pi r v_0}$

(b) If $r = 20$ cm and $v_0 = 0.01$ cm/min

$$\Rightarrow \quad \frac{dA}{dt} = 2\pi(20\text{ cm})(0.01\text{ cm/min})$$

$$\Rightarrow \quad \boxed{\frac{dA}{d\pi} = 0.4\pi \text{ cm}^2/\text{min}}$$

$$= \cancel{1}$$

③ Sketch the graph of the function:

$$f(x) = \frac{3x^2}{x^2-4}$$

(a) Domain: $\text{Dom}(f) = \mathbb{R} \setminus \{2, -2\}$.

(b) Y-intersept: $(x=0)$ $\quad y = f(0) = \frac{3(0)^2}{0^2-4} = 0$

$\Rightarrow y = 0 \quad \Rightarrow (0,0)$

X-intersept $(y=0)$ $\Rightarrow y = f(x) \Rightarrow 0 = f(x)$

$\Rightarrow 0 = \frac{3x^2}{x^2-4} \Rightarrow 0 = 3x^2 \Rightarrow x = 0$

$\Rightarrow (0,0)$

It crosses the origin, $(0,0)$, only.

(c) $f(-x) = \frac{3(-x)^2}{(-x)^2-4} = \frac{3x^2}{x^2-4} = f(x)$

then $f(x)$ is <u>even.</u> Only study $[0,\infty)$.

(d). <u>Asymptotes.</u>

$$\lim_{x\to 2^-} \frac{3x^2}{x^2-4} = 12 \lim_{x\to 2^-} \frac{1}{x^2-4} = -\infty$$

$\Rightarrow$ <u>$x=2$ is a vertical asymptote.</u>

Also:

$$\lim_{x\to 2^+} \frac{3x^2}{x^2-4} = 12 \lim_{x\to 2^+} \frac{1}{x^2-4} = +\infty$$

<u>Now</u>:

$$\lim_{x\to\infty} \frac{3x^2}{x^2-4} = \lim_{x\to\infty} \frac{3}{1-\frac{4}{x^2}} = \frac{3}{1-0} = 3$$

Then $y=3$ is a <u>horizontal asymptote</u>.

(e) To compute the intervals of increasing and decreasing, we need to compute the derivative:

$$f'(x) = \frac{(x^2-4)\,6x - 3x^2(2x)}{(x^2-4)^2} = \frac{6x^3 - 24x - 6x^3}{(x^2-4)^2}$$

i.e.

$$f'(x) = \frac{-24x}{(x^2-4)^2}$$

Since denominator $(x^2-4)^2 > 0$ (since $x \neq \pm 2$),

and since $x \geq 0$ (we are studying $[0, \infty)$).

then, $f(x)$ decreases on $(0, 2)$ and on $(2, \infty)$.

	$(-\infty, -2)$	$(-2, 0)$	$(0, 2)$	$(2, \infty)$
$f(x)$	$\nearrow$	$\nearrow$	$\searrow$	$\searrow$

Now, if $x < 0$, then, $f'(x) > 0 \Rightarrow f \nearrow$ on $x < 0$.

(f) Therefore, we have a local max. at $x = 0$

$$\max_{loc} f(x) = f(0) = 0.$$

There is no global max since $\lim_{x \to 2^+} f(x) = +\infty$.

Also, there is no global min since $\lim_{x \to 2^-} f(x) = -\infty$.

There is no local min either.

(g) We need to compute the 2nd derivative.

$$f''(x) = \frac{(x^2-4)^2(-24) - (-24x)\,2(x^2-4)\,2x}{(x^2-4)^4}$$

$$f''(x) = (-24) \cdot \left[\frac{(x^2-4)^2 - 4x^2(x^2-4)}{(x^2-4)^4} \right] = (-24)\frac{(x^2-4)\left[(x^2-4) - 4x^2\right]}{(x^2-4)^4}$$

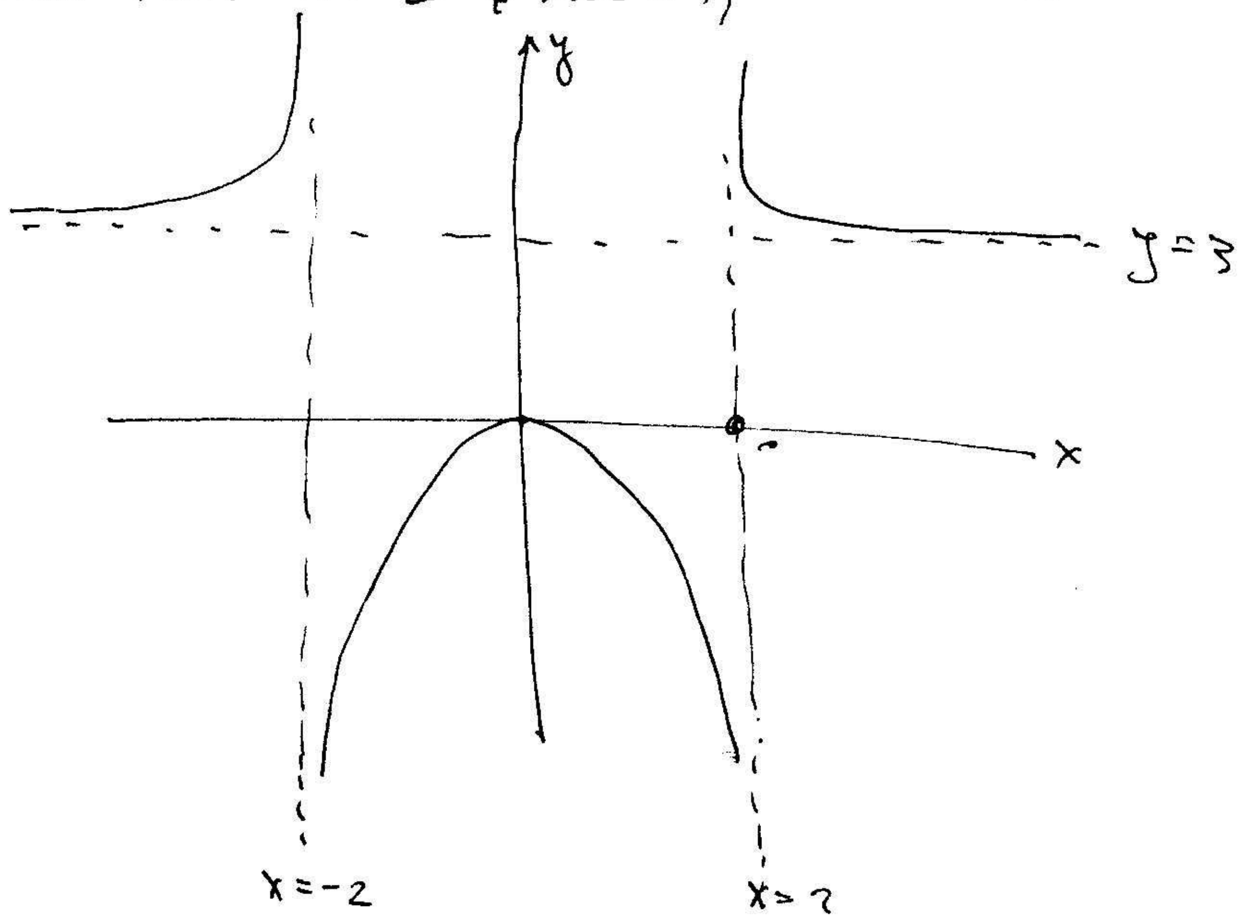
$$= (-24)\left(\frac{-3x^2-4}{(x^2-4)^3} \right) = 24\left(\frac{3x^2+4}{(x^2-4)^3} \right)$$

Notice that the denominator defines de sign of $f''(x)$, since $24(3x^2+4) > 0$.

Hence

	$(0, 2)$	$(2, \infty)$
f''	<0	>0
f	downwards concave	upwards concave.

Since ~~then~~ $x = 2 \notin Dom(f)$, there is no inflexion pts



Critical points.

$f'(x)$ does not exist at $\underline{x=2}$, but this is a vertical asymptote..

$f'(x)=0$ at $x=0$, only

$f''(0) = \frac{24 \cdot 4}{(-4)^3} < 0 \Rightarrow f(0)$ is a local máximum.
