UNIVERSIDAD AUTÓNOMA METROPOLITANA - AZCAPOTZALCO
CÁLCULO DIFERENCIAL
TRIMESTRE: OTOÑO DE 2016.

EXAMEN # 3.
FECHA DE ENTREGA: **VIERNES 9 DE DICIEMBRE DE 2016.**
HORA: **16:00 HORAS.**

Nombre: ANSWER KEY.

Instrucciones:

- El examen consta de **ONCE** problemas: diez problemas 10 puntos cada uno y uno de 20 puntos. Puede obtener hasta un total 120 puntos, a calificar sobre 100.
- **ARGUMENTEN SUS RESPUESTAS. DESARROLLEN SUS CUENTAS. Simplifiquen.** Problema sin argumento o desarrollo vale **CERO** puntos.

PROBLEMAS

(1) **(10 puntos.)** Un granjero necesita encerrar un espacio rectangular junto a un río (recto) con 1200 metros cerca. No necesita poner cerca en el río. Indique las dimensiones del campo que contenga una área máxima.

(2) **(10 puntos.)** Considere la función

$$f(x) = x^2 - 2x + 1$$

- (a) ¿Es f inyectiva? ¿Por qué?
- (b) Si es inyectiva, encuentre f^{-1}. Si no es inyectiva, hágala inyectiva y calcule f^{-1}.
- (c) Dibuje en el mismo plano cartesiano, las gráficas de f y f^{-1}.
- (d) Calcule la derivada de f^{-1} directamente del inciso (b).
- (e) Usando el teorema de la función inversa, calcule la derivada de f^{-1}. Compare con el resultado de (d).

(3) **(10 puntos.)** Resuelva la ecuación

$$e^{x^2} e^{-5x} e^6 = 1.$$

(4) **(10 puntos.)** Calcule la derivada de la función

$$g(t) = \ln(2e^t \cos t).$$

(5) **(10 puntos.)** Calcule la derivada de la función

$$h(x) = (2x)^{\sin 2x}.$$

(6) **(10 puntos.)** Considere la función

$$F(x) = \ln(\ln x).$$

- (a) Encuentre el dominio de F.
- (b) Calcule dF/dx.

(7) **(10 puntos.)** Calcule

$$\lim_{x \to 2\pi} \frac{\ln(\sec x)}{(x - 2\pi)^2}$$

(8) **(10 puntos.)** Calcule la derivada de

$$G(x) = \text{Arctan}(\ln x).$$

(9) **(10 puntos.)** Calcule

$$\lim_{x \to +\infty} x \,\text{Arctan}\left(\frac{2}{x}\right).$$

(10) **(10 puntos.)** Usando un polinomio de Taylor de grado 3, dé la mejor aproximación de $\cos(47^\circ)$.

(11) **(20 puntos extra.)** Considere la función

$$G(x) = \ln(\ln(|\sin x|)).$$

- (a) Encuentre el dominio de G.
- (b) Calcule dG/dx.

UNIVERSIDAD AUTÓNOMA METROPOLITANA – AZCAPOTZALCO

CÁLCULO DIFERENCIAL   Friday, December 9, 2016.

Exam #3.

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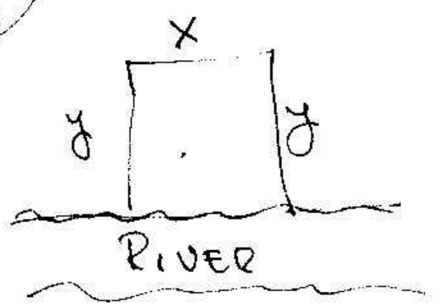
(1)



Perimeter : $P = x + 2y$.

with $P = 1200\,m$.

Area $= A = xy$.

Want to maximize A.

Now, $y = \frac{P-x}{2} \Rightarrow A(x) = x\frac{(P-x)}{2}$

$Dom(A) = [0, P]$ and A is continuous. It reaches its max and min.

$\frac{dA(x)}{dx} = \frac{P-x}{2} + x\left(-\frac{1}{2}\right) = \frac{P-2x}{2}$; $A'(x) = 0 \Rightarrow P - 2x = 0 \Rightarrow x = \frac{P}{2}$

Critical points $x = 0,\ x = P$ (boundary points).

$A'(x)$ always exists

$A'(x) = 0 \ :\ x = \frac{P}{2}$.

If $0 < x < \frac{P}{2} \Rightarrow 2x < P \Rightarrow 0 < P - 2x$

$\Rightarrow A'(x) = \frac{P-2x}{2} > 0 \Rightarrow A \nearrow$ in $(0, \frac{P}{2})$.

If $\frac{P}{2} < x < P \Rightarrow P < 2x \Rightarrow P - 2x < 0$

$\Rightarrow A'(x) = \frac{P-2x}{2} < 0 \Rightarrow A \searrow$ in $(\frac{P}{2}, P)$.

Hence, $A(\frac{P}{2})$ is the global maximum:

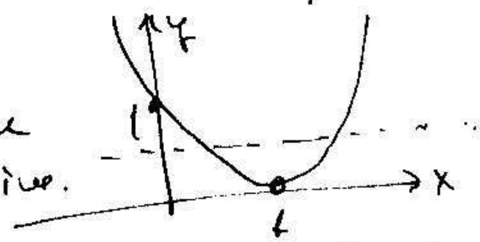
$$A\left(\frac{P}{2}\right) = \frac{P}{2}\left(\frac{P - \frac{P}{2}}{2}\right) = \frac{P}{2}\left(\frac{P/2}{2}\right) = \frac{P^2}{8} = \frac{(1200\,m)^2}{8} = 180{,}000\ m^2$$

Dimensions $\boxed{x = \frac{P}{2} = 600\,m} \Rightarrow y = \frac{P - \frac{P}{2}}{2} = \frac{P/2}{2} = \frac{P}{4}$.

$\boxed{y = \frac{P}{4} = 300\,m}$

(2) $f(x) = x^2 - 2x + 1 = (x-1)^2$.

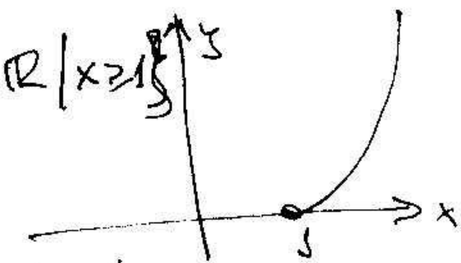
(a) f no es inyectiva, la gráfica es la curva $y = (x-1)^2$, la cual es una parabola que abre hacia arriba desplazada 1 unidad a la derecha:



Since there is a horizontal line crossing twice, it is not injective.

(b) To make it injective, let us restrict its domain to:

$$Dom(f) = [1, \infty) = \{x \in \mathbb{R} \mid x \geq 1\}$$



Then, it is injective.

To compute its inverse, we start at:

(1) $y = (x-1)^2$

(2) and solve for x:

$$\pm\sqrt{y} = x - 1$$

$$\pm\sqrt{y} + 1 = x$$

$$x = \pm\sqrt{y} + 1$$

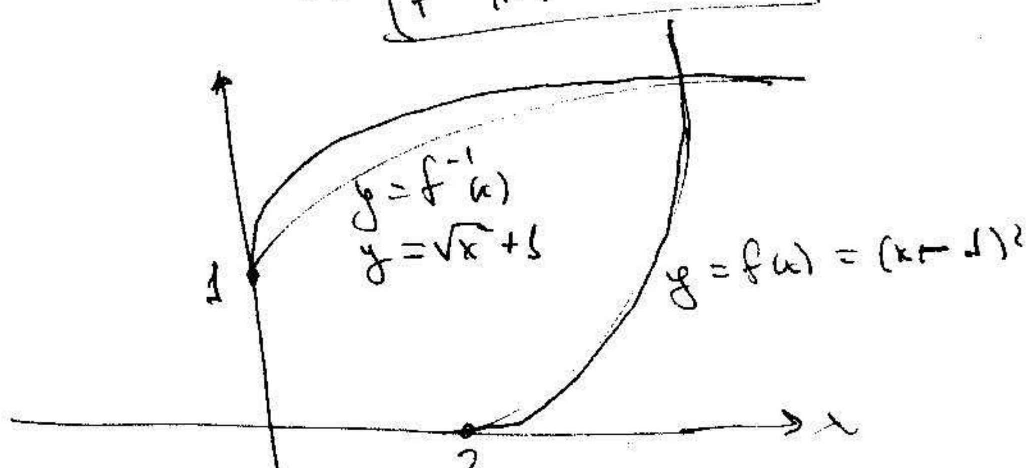
Switching $x \leftrightarrow y$

Since $x \geq 0$, we should take "+".

$$\Rightarrow \quad y = +\sqrt{x} + 1$$

$$\Rightarrow \quad \boxed{f^{-1}(x) = \sqrt{x} + 1}$$ is the inverse function

(c)



(d) $\frac{df^{-1}}{dx} = \frac{d}{dx}(\sqrt{x}+1) = \frac{1}{2\sqrt{x}}$.

(e) $f(y) = (y-1)^2$ is my original function with $\frac{df}{dy} = 2(y-1)$

$y = f^{-1}(x) = \sqrt{x}+1$ is the inverse function.

This way:

$$\frac{df^{-1}}{dx}(x) = \frac{1}{\frac{df}{dy}\Big|_{y=f^{-1}(x)}} = \frac{1}{2(y-1)\Big|_{y=f^{-1}(x)}} = \frac{1}{2(f^{-1}(x)-1)}$$

$$= \frac{1}{2((\sqrt{x}+1)-1)} = \frac{1}{2(\sqrt{x})}$$ ✓ Same function.

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(3) We have: $e^{x^2} e^{-5x} e^{6} = 1$

Using the properties of the exponents:

$$e^{x^2-5x+6} = 1.$$

By definition of $\ln(x)$:

$$x^2 - 5x + 6 = \ln(1)$$

i.e. $x^2 - 5x + 6 = 0$

$$(x-3)(x-2) = 0$$

$$\Rightarrow \boxed{x_1 = 2} \quad \boxed{x_2 = 3}$$

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(4) Use the properties of $\ln(x)$:

$$g(t) = \ln 2 + \ln(e^t) + \ln(\cos t)$$

$$= \ln 2 + t + \ln(\cos t)$$

$$\Rightarrow \frac{dg}{dt} = 0 + 1 + \frac{d}{dt}\ln(\cos t) = 1 + \frac{(\cos t)'}{\cos t} \Rightarrow \boxed{\frac{dg}{dt} = 1 - \frac{\sin t}{\cos t}}$$

$$\boxed{g' = 1 - \tan t}$$

(5) We use logarithmic differentiation:

$h(x) = (2x)^{\sin(2x)} \Rightarrow \ln(h(x)) = \ln\left((2x)^{\sin(2x)}\right)$

i.e. $\ln(h(x)) = \sin(2x)\ln(2x)$

Computing derivatives:

$$\frac{d}{dx}\ln(h(x)) = \frac{d}{dx}\left(\sin(2x)\ln(2x)\right).$$

By the chain rule, product rule and derivatives of $\ln(x)$:

$$\frac{1}{h(x)}\cdot\frac{dh}{dx} = (\sin(2x))'\ln(2x) + \sin(2x)(\ln(2x))'$$

$$= 2\cos 2x \ln(2x) + \sin(2x)\frac{2}{2x}$$

i.e. $\frac{1}{h(x)}\frac{dh(x)}{dx} = 2\cos 2x\ln(2x) + \frac{\sin(2x)}{x}$

i.e. $\boxed{\frac{dh}{dx} = h(x)\left(2\cos(2x)\ln(2x) + \frac{\sin(2x)}{x}\right)}$

or $\boxed{\frac{dh}{dx} = (2x)^{\sin(2x)}\left(2\cos(2x)\ln(2x) + \frac{\sin(2x)}{x}\right)}$

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(6) Let: $F(x) = \ln(\ln(x))$.

(a) Dom F.

$\ln(y)$ is defined for $y > 0$, so we require $\ln x > 0$.

To have $\ln(x) > 0 \Rightarrow x > 1$

Hence: $\boxed{\text{Dom}(F) = (1, \infty).}$

(b) $\frac{dF}{dx} = \frac{d}{dx}\ln(\ln(x)) = \frac{1}{\ln(x)}\cdot\frac{d}{dx}(\ln x) = \frac{1}{\ln(x)}\cdot\frac{1}{x}$

i.e. $\boxed{\frac{dF}{dx} = \frac{1}{x\ln x}}$

⑦ Compute:

$$\lim_{x\to 2\pi} \frac{\ln(\sec x)}{(x-2\pi)^2}$$

Direct substitution implies:

$$\ln(\sec(2\pi)) = \ln(1/\cos(2\pi)) = \ln(1/1) = 0$$

and

$$(x-2\pi)^2\Big|_{x=2\pi} = (2\pi-2\pi)^2 = 0$$

Then, we would have an undetermined limit $\left(\frac{0}{0}\right)$.

Make the change of variable.

$$x = y + 2\pi$$

Then:
$$\sec(x) = \frac{1}{\cos x} = \frac{1}{\cos(y+2\pi)} = \frac{1}{\cos y} = \sec(y).$$

$$(x-2\pi)^2 = y^2.$$

and
$$x \longrightarrow 2\pi$$
$$y+2\pi \longrightarrow 2\pi$$
$$y \longrightarrow 0.$$

Hence:
$$\lim_{x\to 2\pi} \frac{\ln(\sec x)}{(x-2\pi)^2} = \lim_{y\to 0} \frac{\ln(\sec(y))}{y^2}$$

Still we have "$\frac{0}{0}$": $\ln(\sec(0)) = \ln(1) = 0$

$$y^2\Big|_{y=0} = 0.$$

But we can use L'Hôpital:

$$\lim_{y\to 0} \frac{\ln(\sec y)}{y^2} \stackrel{\downarrow}{=} \lim_{y\to 0} \frac{\frac{1}{(\sec y)}\cdot(\sec y)'}{(y^2)'} = \lim_{y\to 0} \frac{\frac{\sec y \tan y}{\sec y}}{2y}$$

$= \lim_{y\to 0} \frac{\tan y}{2y}$ is still undetermined of the form "$\frac{0}{0}$"

Use L'Hôpital again:

$$= \lim_{y\to 0} \frac{(\tan y)'}{(2y)'} = \lim_{y\to 0} \frac{1+\tan^2 y}{2} = \frac{1+0}{2} = \frac{1}{2}$$

i.e. $$\boxed{\lim_{x\to 2\pi} \frac{\ln(\sec x)}{(x-2\pi)^2} = \frac{1}{2}}$$

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(8) Compute the derivative of

$$G(x) = \text{Arctan}(\ln x)$$

$$\frac{dG}{dx} = \frac{d}{dx}\text{Arctan}(\ln x) = \frac{1}{1+(\ln x)^2}\cdot\frac{d(\ln x)}{dx}$$

$$= \frac{1}{1+(\ln x)^2}\cdot\frac{1}{x}$$

i.e. $$\boxed{\frac{d}{dx}\text{Arctan}(\ln x) = \frac{1}{x\,(1+x^2)}}$$

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(9) Compute: $\lim_{x\to\infty} x\,\text{Arctan}\left(\frac{2}{x}\right)$.

It is of the form "$0\cdot\infty$".

Take $y = \frac{1}{x} \Rightarrow y \xrightarrow[x\to\infty]{} 0$, thence

$\lim_{x\to\infty} x\,\text{Arctan}\left(\frac{2}{x}\right) = \lim_{y\to 0}\frac{\text{Arctan}(2y)}{y}$ is of the same "$\frac{0}{0}$". Use L'Hôpital:

$$= \lim_{y\to 0}\frac{(\text{Arctan}(2y))'}{(y)'} = \lim_{y\to 0}\frac{1}{1+(2y)^2}(2) = 2$$

$$\Rightarrow \boxed{\lim_{x\to\infty} x\,\text{Arctan}\left(\frac{2}{x}\right) = 2}$$

(10) Approximate $\cos(47°)$ using a 3rd degree Taylor polynomial. We require, to compute

$f(x) = \cos x$ at $x = \frac{\pi}{4}$. $f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$f'(x) = -\sin x$ (since $\frac{\pi}{4} = 45°$ is close to $47°$) $f'\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$f''(x) = -\cos x$ $f''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$f'''(x) = +\sin x$ $f'''\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$.

Then, for x close to $a = \frac{\pi}{4}$:

$$f(x) \approx P_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{1}{3!}f'''(a)(x-a)^3$$

$$= f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right) + \frac{1}{2}f''\left(\frac{\pi}{4}\right)(x-a)^2 + \frac{1}{6}f'''\left(\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right)^3$$

$$= \frac{\sqrt{2}}{2}\left(1 - \left(x-\frac{\pi}{4}\right) - \frac{1}{2}(x-a)^2 + \frac{1}{6}\left(x-\frac{\pi}{4}\right)^3\right).$$

Now: $x - \frac{\pi}{4} = x - 45° = 47° - 45° = 2° = \frac{\pi}{90}$. $\Rightarrow$ $\boxed{x = \frac{\pi}{4} + \frac{\pi}{90}}$

Hence,

$$\boxed{\cos(47°) \approx P_3(x) = \frac{\sqrt{2}}{2}\left(1 - \frac{\pi}{90} - \frac{1}{2}\left(\frac{\pi}{90}\right)^2 + \frac{1}{6}\left(\frac{\pi}{90}\right)^3\right).}$$

$\approx 0.6819983166 2718.$

Directly from your calculator:

$\cos(47°) \approx 0.681998360062499$

Error $= 5 \times 10^{-8}$