Cálculo Diferencial. Evaluación global (trimestre 16-O), turno vespertino

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El global consta de los ejercicios con **.
Todas las respuestas necesitan desarrollo o justificación.

Primer parcial

1. Derivar las funciones:

   a). **(** 10 puntos)** $g(x) = [\operatorname{sen}^2(x) + \cos(x^2)]^7$. b). **(** 5 puntos)** $h(x) = \dfrac{4-x^2}{4+x^2}$.

2. **(** 10 puntos)** Encuentre la ecuación de la recta tangente al lugar geométrico definido por

   $y = \tan(xy)$,

   en el punto $(\pi/4, 1)$.

3. **(** 10 puntos)** Un globo se eleva verticalmente con una velocidad de 5 m/s desde el nivel del piso y a 60 m de un observador. Determine la velocidad con la que se aleja del observador cuando tiene una altura de 40 m.

4. La posición de un móvil en todo instante t es:

$$s(t) = t^3 - 6t^2 + 9t \text{ m}.$$

   a. Determinar la aceleración cuando su velocidad es cero.
   b. ¿Cuándo se desplaza para adelante y cuándo para atrás?
   c. ¿Cuál es su posición cuando su aceleración es cero?

Segundo parcial

1. Encontrar los extremos absolutos de la función $f(x) = \operatorname{sen} x + \cos x$ en el intervalo $[0, \pi]$.

2. **(** 15 puntos)** Sea la función:

$$f(x) = \frac{x^2 - 5}{9 - x^2}.$$

   Proporcionar:

   a. Dominio y raíces de esa función.
   b. Asíntotas horizontales y verticales.
   c. Puntos críticos. Intervalos de monotonía.
   d. Puntos de inflexión. Intervalos de concavidad.
   e. Un bosquejo de la gráfica.

3. **(** 15 puntos)** Diseñe un contenedor cilíndrico de superficie mínima que tenga tapa y una capacidad de 50 metros cúbicos.

Tercer parcial

1. Determine un dominio donde la función $f(x) = \sqrt{x^2 - 4}$ tenga función inversa. Proporcione la expresión que define a dicha función inversa.

2. Derivar las funciones:

   a). **(** 5 puntos)** $f(x) = 2^{\sqrt{\ln x}}$. b). **(** 10 puntos)** $f(x) = x^{\frac{1}{x^2-1}}$.

3. Calcular los siguientes límites usando la regla de L'Hôpital:

   a). $\lim_{x\to 0} \left(\frac{x - \operatorname{sen} x}{\tan x - \operatorname{sen} x} \right)$. b). **(** 10 puntos)** $\lim_{x\to 0} \left(\frac{1}{\operatorname{sen} x} - \frac{1}{x} \right)$.

4. **(** 10 puntos)** Encuentre el polinomio de Taylor, de grado 3, de la función $y = x \ln x$, alrededor del punto $c = 1$.

Universidad Autónoma Metropolitana - Azcapotzalco.

Cálculo Diferencial — Martes, 13 de diciembre de 2016

Examen Global.

PART I

(1) (a) $\frac{dg}{dx} = \frac{d}{dx}\left(\sin^2(x) + \cos(x^2)\right)^7$

$= 7\left(\sin^2(x) + \cos(x^2)\right)^6 \frac{d}{dx}\left(\sin^2(x) + \cos(x^2)\right)$

Chain rule

$= 7\left(\sin^2(x) + \cos(x^2)\right)^6 \left(2\sin x \frac{d}{dx}(\sin x) - \sin(x^2)\frac{d}{dx}(x^2)\right)$

Chain rule

$= 7\left(\sin^2(x) + \cos(x^2)\right)^6 \left(2\sin x\cos x - 2x\sin(x^2)\right)$

trig. derivatives and polynomials.

$$\boxed{\frac{dg}{dx} = 7\left(\sin^2(x) + \cos(x^2)\right)^6\left(2\sin x\cos x - 2x\sin(x^2)\right)}$$

Quotient rule.

(b) $\frac{dh}{dx} = \frac{d}{dx}\left(\frac{4-x^2}{4+x^2}\right) = \frac{(4+x^2)(4-x^2)' - (4-x^2)(4+x^2)'}{(4+x^2)^2}$

$= \frac{-2x(4+x^2) - 2x(4-x^2)}{(4+x^2)^2} = \frac{-8x - 2x^3 - 8x + 2x^3}{(4+x^2)^2}$

$\Rightarrow$ $$\boxed{\frac{dh}{dx} = \frac{-16x}{(4+x^2)^2}}$$

(2) $y = \tan(xy)$.

We require implicit differentiation.

Notice that $\left(\frac{\pi}{4}, 1\right)$ solves the equation:

$$1 = \tan\left(\frac{\pi}{4} \cdot 1\right) \checkmark$$

Now, differentiating, assuming $y = y(x)$:

$$\frac{dy}{dx} = \frac{d}{dx}\left(\tan(xy)\right) = \left(1 + \tan^2(xy)\right)\frac{d}{dx}(xy)$$

$$= \left(1 + \tan^2(xy)\right)\left(x'y + xy'\right)$$

$$= \left(1 + \tan^2(xy)\right)\left(y + x\frac{dy}{dx}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} - \left(1 + \tan^2(xy)\right)x\frac{dy}{dx} = \left(1 + \tan^2(xy)\right)y$$

i.e. $$\left(1 - \left(1 + \tan^2(xy)\right)x\right)\frac{dy}{dx} = \left(1 + \tan^2(xy)\right)y$$

$$\Rightarrow \quad \boxed{\frac{dy}{dx} = \left(\frac{1 + \tan^2(xy)}{1 - \left(1 + \tan^2(xy)\right)x}\right)y}$$

Evaluating at $\left(\frac{\pi}{4}, 1\right)$, we get the slope of the tangent line

$$\frac{dy}{dx} = \frac{\left(1 + \tan^2\left(\frac{\pi}{4}\right)\right)}{\left(1 - \left(1 + \tan^2\frac{\pi}{4}\right)\frac{\pi}{4}\right)} \cdot 1 = \frac{(1+1)}{\left(1 - (1+1)\frac{\pi}{4}\right)} = \frac{2}{1 - \frac{\pi}{2}}$$

$$\Rightarrow \quad \underline{m = \frac{4}{2 - \pi}}$$

And the equation of the tangent line at $\left(\frac{\pi}{4}, 1\right)$ is

$$\boxed{y - 1 = \frac{4}{2 - \pi}\left(x - \frac{\pi}{4}\right)}$$

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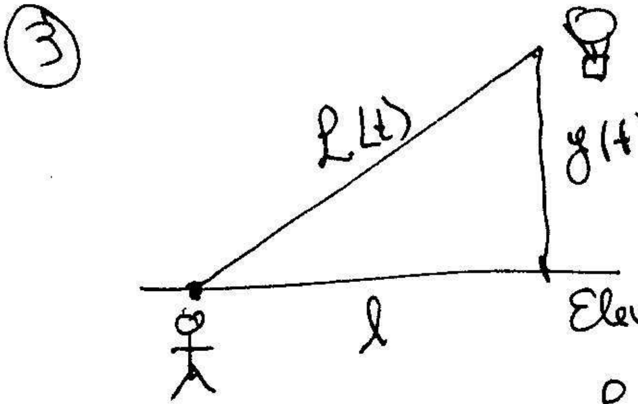


$l = 60\,m$

$y(t)$ is a function of time.

Elevation velocity $= \dot{y}(t) = 5$ meters/sec

$L(t)$ is the distant from the globe to the observer, and it is a function of time

$$L(t) = \sqrt{l^2 + y^2(t)}.$$

Now, the globe goes away from observer at a velocity $\dot{L}(t)$, i.e..

$$\dot{L}(t) = \frac{(l^2 + y^2(t))^{\bullet}}{2\sqrt{l^2 + y^2(t)}} = \frac{2\,y(t)\,\dot{y}(t)}{2\sqrt{l^2+y^2(t)}}$$

i.e. $$\boxed{\dot{L}(t) = \frac{y(t)\,\dot{y}(t)}{\sqrt{l^2+y^2(t)}}}$$

At same time, $y(t) = 40$. Thus:

$$\dot{L} = \frac{40\cdot 5}{\sqrt{60^2 + 40^2}} = \frac{40\cdot 5}{10\sqrt{6^2+4^2}} = \frac{40\cdot 5}{10\sqrt{36+16}}$$

$$= \frac{40\cdot 5}{10\sqrt{52}} = \frac{4\cdot 5}{\sqrt{52}} = \frac{40}{\sqrt{52}}$$

$\boxed{\dot{L} = \frac{40}{\sqrt{52}}\ m/sec}$ or, approximately $\boxed{\dot{L} \approx 7.55\ m/sec}$

(4) (a) $s(t) = t^3 - 6t^2 + 9t = t(t-3)^2$

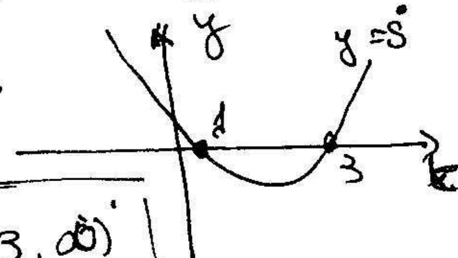
Velocity: $\dot{s}(t) = 3t^2 - 12t + 9 = 3(t^2 - 4t + 3)$
$= 3(t-1)(t-3)$

Acceleration $\ddot{s}(t) = 6t - 12 = 6(t-2)$.

Velocity $= 0 \Rightarrow \dot{s} = 0 \Rightarrow 3(t-1)(t-3) = 0$
$\Rightarrow$ at $t = 1$ and $t = 3$.

Now $\boxed{\ddot{s}(1) = 6(1-2) = -6\ m/sec^2}$
$\boxed{\ddot{s}(3) = 6(3-2) = 6\ m/sec^2}$

(b) Adelante Forward displacement



$\dot{s} > 0$: $3(t-1)(t-3) > 0$.
when $\boxed{t \in (-\infty, 1) \cup (3, \infty)}$

Backwards $\dot{s} < 0$, $3(t-1)(t-3) < 0$
$\Rightarrow \boxed{t \in (1,3)}$

(c). $\ddot{s}(t) = 0$ when $t = 2$
$6(t-2) = 0$

Then $s(2) = 2(2-3)^2 = 2m$.

$\boxed{s(2) = 2}$

SEGUNDA PARTE.

(1) $f(x) = \sin x + \cos x$. is a continuous function in a closed and bounded interval $[0,\pi]$. Hence, it reaches its extremum values.

$f'(x) = \cos x - \sin x$.

Critical points: $x=0$, $x=\pi$, boundary values.

$f'(x)$ always exist.

$f'(x) = 0$ when $\cos x - \sin x = 0$

Now: $\cos x = \sin x$ when $x = \frac{\pi}{4}$ and $x = \frac{5}{4}\pi$,

but $\frac{5}{4}\pi \notin [0,\pi]$.

Then, critical points $\boxed{x=0,\ x=\frac{\pi}{4},\ x=\pi}$

$f(0) = \sin(0) + \cos 0 = 0 + 1 = 1$

$f(\frac{\pi}{4}) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$.

$f(\pi) = \sin\pi + \cos\pi = 0 - 1 = -1$

$\Rightarrow \boxed{\begin{array}{l} \max\limits_{[0,\pi]}(f) = \sqrt{2} \\ \min\limits_{[0,\pi]}(f) = -1 \end{array}}$

Notice that, on $[0,\frac{\pi}{4}]$, $f'(x) = \cos x - \sin x \geq 0 \Rightarrow f \nearrow$

and on $[\frac{\pi}{4},\pi]$, $f'(x) = \cos x - \sin x \leq 0 \Rightarrow f \searrow$

$\Rightarrow$ max at $x = \frac{\pi}{4}$, thus $\max(f) = f(\frac{\pi}{4}) = \sqrt{2}$

(2) $f(x) = -\left(\frac{x^2-5}{x^2-9}\right) = -\frac{(x-\sqrt{5})(x+\sqrt{5})}{(x-3)(x+3)}$

(a) $\mathrm{Dom}(f) = \mathbb{R} \setminus \{3, -3\}$.

Roots $= \{\sqrt{5}, -\sqrt{5}\}$

(b) Since: $\lim_{x \to 3^{\pm}} -\left(\frac{x^2-5}{x^2-9}\right) = \mp \infty$. $\Rightarrow$ $\boxed{x=3}$ vertical asymptote

and

$\lim_{x \to -3^{\pm}} -\left(\frac{x^2-5}{x^2-9}\right) = \pm \infty \Rightarrow \boxed{x=-3}$ is a vertical asymptote.

Now;

$$\lim_{x \to \pm\infty} -\frac{(x^2-5)}{(x^2-9)} = \lim_{x \to \pm\infty} -\frac{(1-5/x^2)}{(1-9/x^2)} = -1$$

$\boxed{y=-1}$ is a horizontal asymptote

(c) $$f'(x) = -\frac{(x^2-9)(x^2-5)' - (x^2-5)(x^2-9)'}{(x^2-9)^2}$$

$$= -\frac{2x(x^2-9) - 2x(x^2-5)}{(x^2-9)^2}$$

$$= -\frac{2x^3 - 18x - 2x^3 + 10x}{(x^2-9)^2}$$

$$\boxed{f'(x) = \frac{8x}{(x^2-9)^2}}$$

Critical points There is no boundaries

$f'(x)$ does not exist at $x = \pm 3$.

$f'(x) = 0$ at $x = 0$

Critical point $= \{0, +3, -3\}$.

Notice that $f'(x) = \frac{8x}{(x^2-9)^2} > 0$, if $x > 0$ $(x \neq 3)$

and $f'(x) = \frac{8x}{(x^2-9)} < 0$, if $x < 0$ $(x \neq -3)$

Then: $f \searrow$ in $(-\infty, -3)$ and $(-3, 0)$

and $f \nearrow$ in $(0, 3)$ and $(3, 0)$.

(d) $$f''(x) = \frac{(x^2-9)^2(8x)' - 8x\left((x^2-9)^2\right)'}{(x^2-9)^4}$$

$$= \frac{8(x^2-9)^2 - 8x\left(2(x^2-9)2x\right)}{(x^2-9)^4}$$

$$= 8(x^2-9)\frac{\left((x^2-9) - 4x^2\right)}{(x^2-9)^4}$$

$$= 8\,\frac{-3x^2-9}{(x^2-9)^3}$$

$$= -8\,\frac{(3x^2+9)}{(x^2-9)^3}$$

Notice that

$8(3x^2+9) > 0$

Then

$-\frac{1}{(x^2-9)^3}$ defines

the sign of f''.

If $|x| \geq 3$; i.e. ~~x~~ 3

$x \leq -3$ and $3 \leq x$.

then $f'' < 0 \Rightarrow f$ is concave down.

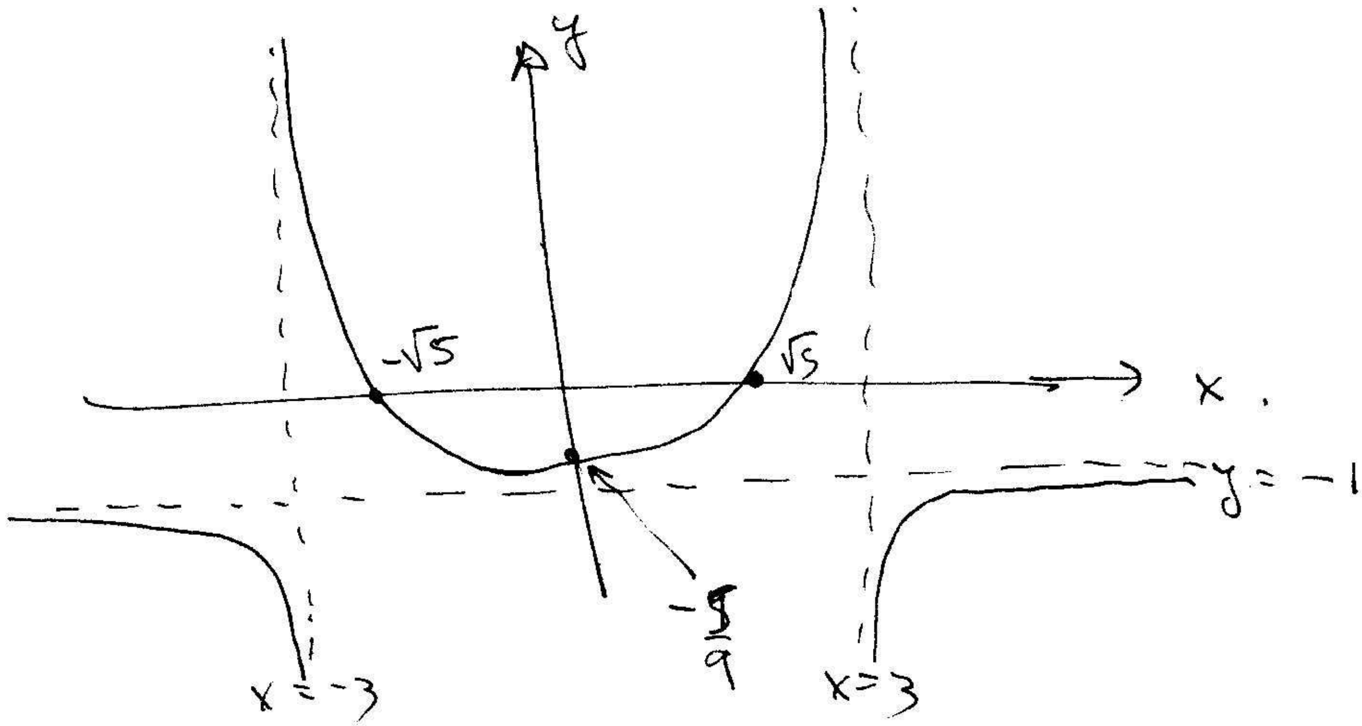
①

If $|x| \leq 3 \Rightarrow -3 \leq x \leq 3$

$\Rightarrow f'' \geq 0 \Rightarrow f$ is concave up on $(-3,3)$

②

(e) The graph of the function is



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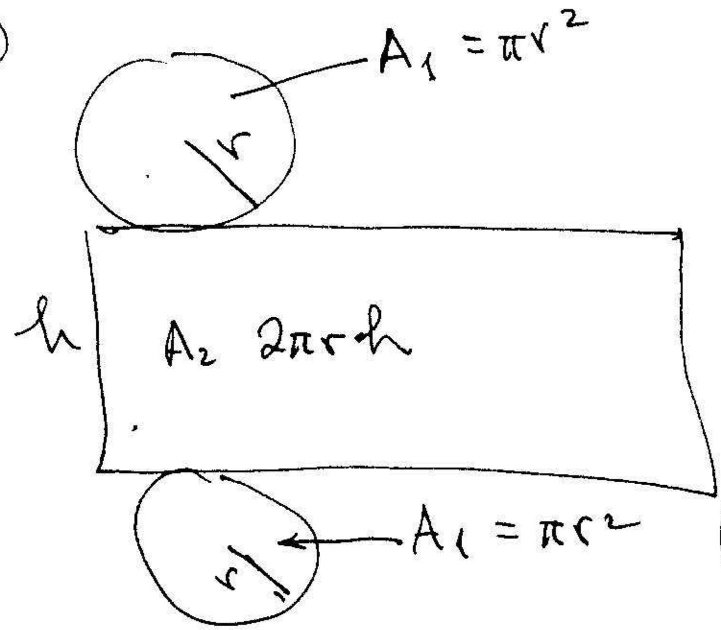


Total surface area:

$$A = 2\pi r^2 + 2\pi r h.$$

Volume: $V_0 = \pi r^2 h.$

(with $V_0 = 80\ m^3$)

Now $h = \dfrac{V_0}{\pi r^2}$

$$\Rightarrow \quad A(r) = 2\pi r^2 + 2\pi r\left(\frac{V_0}{\pi r^2}\right)$$

ie

$$\boxed{A(r) = 2\pi r^2 + \frac{2V_0}{r}.}$$

<u>Domain</u>: $r \geq 0$, but $r \neq 0$.

$\mathrm{Dom}(A) = (0, \infty).$

Now: $A'(r) = 4\pi r - \dfrac{2V_0}{r^2}.$

<u>Critical points</u>. There is <u>no</u> boundaries,

$A'(r)$ does <u>not</u> exists in $r=0$, but

$r = 0 \notin \mathrm{Dom}(A)$

$A'(r) = 0$, at. $\boxed{r_0 = \sqrt[3]{\dfrac{V_0}{2\pi}} = \sqrt[3]{\dfrac{25}{\pi}}}$

Now:

If $0 < r < \sqrt[3]{\frac{V_0}{2\pi}} = r_0$

$\Rightarrow \quad r^3 < \frac{V_0}{2\pi} \Rightarrow 4\pi r^3 < 2V_0.$

$\Rightarrow \quad 4\pi r^3 - 2V_0 < 0$

$A'(r) = 4\pi r - \frac{2V_0}{r^2} < 0$

$\Rightarrow A'(r) < 0 \Rightarrow A(r) \searrow 0$ on $(0, r_0)$

If $r_0 = \sqrt[3]{\frac{V_0}{2\pi}} < r$

$\Rightarrow \quad \frac{V_0}{2\pi} < r^3 \Rightarrow 2V_0 < 4\pi r^3$

$\Rightarrow \quad 0 < 4\pi r^3 - 2V_0$

$\Rightarrow \quad 0 < 4\pi r - \frac{2V_0}{r^2} = A'(r)$

$\Rightarrow \quad A'(r) > 0 \Rightarrow A'(r) \nearrow$ on (r_0, ∞)

Then $A(r_0)$ is a global ~~max~~ minimum.

The dimensions should be $\boxed{r_0 = \sqrt[3]{\frac{25}{\pi}} \approx 2m.}$

and $h_0 = \frac{V_0}{\pi r_0^2} = \frac{50}{\pi}\left(\frac{\pi}{25}\right)^{2/3} = 2 \cdot \frac{25}{\pi}\left(\frac{\pi}{25}\right)^{2/3} = 2\left(\frac{25}{\pi}\right)^{1/3}$

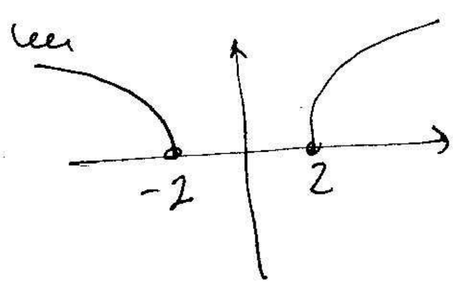
$h_0 = 2\sqrt[3]{\frac{25}{\pi}} = \boxed{h_0 = 2r_0}$ i.e. the hight should be twice the radius.

$\boxed{r_0 \approx 1.99\, m}$ $\boxed{h_0 \approx 3.98 m}$

TERCERA PARTE.

(1) $f(x) = \sqrt{x^2 - 4}$. $\mathrm{Dom}(f) = (-\infty, -2] \cup [2, \infty)$.
$\mathrm{Rn}(f) = [0, \infty)$

The graph has the equation $y = \sqrt{x^2 - 4}$.
Notice that y should be ≥ 0. Then



$$y^2 - x^2 = -4.$$

is a hyperbola, and the graph of $f(x)$ is half a hyperbola.

Restrict the domain to

$\mathrm{Dom}(f) = [2, \infty)$. (and $\mathrm{Rn}(f) = [0, \infty)$).

Then $f(x)$ is 1-1 and has an inverse.

From $\quad y = \sqrt{x^2 - 4}, \quad x \in [2, \infty)$

$$y^2 = x^2 - 4$$

$$y^2 + 4 = x^2$$

$$\pm\sqrt{y^2 + 4} = x$$

Since $x \in [2, \infty) \Rightarrow x > 0 \Rightarrow x = \sqrt{y^2 + 4}$.

Switch $x \leftrightarrow y \Rightarrow y = \sqrt{x^2 + 4}$ and the inverse function is:

$$\boxed{f^{-1}(x) = \sqrt{x^2 + 4}}.$$

Since $y = f(x) \geq 0 \Rightarrow f^{-1}(x) = \sqrt{x^2 + 4} \geq 2$ with $x \geq 0$

$$\boxed{\begin{array}{l} \mathrm{Dom}(f^{-1}) = [0, \infty) \\ \mathrm{Rn}(f^{-1}) = [2, \infty) \end{array}}$$

(2) (a) $f(x) = 2^{\sqrt{\ln x}}$.

Take log on both sides

$$\ln(f) = \ln 2^{\sqrt{\ln x}}$$

i.e. $\ln(f) = \sqrt{\ln x}\,\ln 2$.

Computing derivatives.

$$\frac{d}{dx}\left(\ln(f)\right) = \frac{d}{dx}\left(\ln 2\sqrt{\ln x}\right)$$

$$\frac{f'}{f} = (\ln 2)\,\frac{(\ln x)'}{2\sqrt{\ln x}}$$

$$\Rightarrow \quad \frac{f'}{f} = (\ln 2)\,\frac{1/x}{2\sqrt{\ln x}}$$

$$\Rightarrow f' = (\ln 2)\frac{f(x)}{2x\sqrt{\ln x}} \Rightarrow \boxed{f'(x) = \frac{(\ln 2)\,2^{\sqrt{\ln x}}}{2x\sqrt{\ln x}}}$$

(b) $f(x) = x^{\left(\frac{1}{x^2-1}\right)}$.

Compute Log's:

$$\ln(f) = \ln x^{\left(\frac{1}{x^2-1}\right)} = \frac{1}{x^2-1}\ln x.$$

Compute derivatives:

$$\frac{d}{dx}\left(\ln(f)\right) = \frac{d}{dx}\left(\frac{\ln x}{x^2-1}\right)$$

$$\Rightarrow \frac{f'}{f} = \frac{(x^2-1)(\ln x)' - (\ln x)(x^2-1)'}{(x^2-1)^2} = \frac{(x^2-1)\frac{1}{x} - 2x\ln x}{(x^2-1)^2}$$

$$\Rightarrow \boxed{f'(x) = x^{\frac{1}{x^2-1}} \cdot \frac{(x^2-1) - 2x^2\ln x}{x(x^2-1)^2}}$$

(3) Calculate the limits:

(a) $\lim_{x\to 0} \frac{x-\sin x}{\tan x-\sin x}$ is of the form "$\frac{0}{0}$".

$\overset{L'H}{=} \lim_{x\to 0} \frac{1-\cos x}{\sec^2 x-\cos x}$ is also of the form "$\frac{0}{0}$",

$= \lim_{x\to 0} \frac{\sin x}{2\sec^2 x \tan x+\sin x}$ is still of the form "$\frac{0}{0}$".

$\overset{L'H}{=} \lim_{x\to 0} \frac{\cos x}{2(2\sec^2 x\tan x)\tan x+2\sec^2 x(\sec^2 x)+\cos x}$

$= \lim_{x\to 0} \frac{\cos x}{4\sec^2 x\tan^2 x+2\sec^4 x+\cos x}$

$= \lim_{x\to 0} = \frac{1}{4\cdot 1\cdot 0+2\cdot 1+1} = \frac{1}{0+2+1}$

$\Rightarrow \boxed{\lim_{x\to 0} \frac{x-\sin x}{\tan x-\sin x} = \frac{1}{3}}$

Different form without L'Hôpital rule.

$\frac{x-\sin x}{\tan x-\sin x} = \frac{x-\sin x}{\sin x\left(\frac{1}{\cos x}-1\right)} = \frac{x-\sin x}{\frac{\sin x}{\cos x}(1-\cos x)} = \frac{\frac{1}{3!}x^3-\frac{1}{5!}x^5+\cdots}{\tan x\left(\frac{x^2}{2!}-\frac{x^4}{4!}+\cdots\right)}$

$= \frac{x^3\left(\frac{x^3}{3!}-\frac{1}{5!}x^5+\cdots\right)}{\left(x+\frac{1}{3}x^3+\cdots\right)\left(\frac{x^2}{2!}-\frac{x^4}{4!}+\cdots\right)} = \frac{x^3\left(\frac{1}{3!}-\frac{1}{5!}x^2+\cdots\right)}{x\cdot x^2\left(1+\frac{1}{3}x^2+\cdots\right)\left(\frac{1}{2!}-\frac{x^2}{4!}+\cdots\right)}$

$= \frac{\left(\frac{1}{3!}-\frac{1}{5!}x^2+\cdots\right)}{\left(1+\frac{1}{3}x^2+\cdots\right)\left(\frac{1}{2!}-\frac{x^2}{4!}+\cdots\right)} \xrightarrow[x\to 0]{} \frac{\frac{1}{3!}}{(1)\left(\frac{1}{2!}\right)} = \frac{2!}{3!} = \frac{1}{3}$

$\Rightarrow \boxed{\lim_{x\to 0} \frac{x-\sin x}{\tan x-\sin x} = \frac{1}{3}}$

(b) $\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right) = \lim_{x\to 0}\frac{x-\sin x}{x\sin x}$ is of the form "$\frac{0}{0}$"

$$\overset{L'H}{=} \lim_{x\to 0}\frac{(x-\sin x)'}{(x\sin x)'} = \lim_{x\to 0}\frac{1-\cos x}{\sin x + x\cos x}$$

is again of the form "$\frac{0}{0}$".

$$\overset{L'H}{=} \lim_{x\to 0}\frac{(1-\cos x)'}{(\sin x + x\cos x)'} = \lim_{x\to 0}\frac{+\sin x}{\cos x + \cos x - x\sin x}$$

$$= \lim_{x\to 0}\frac{\sin x}{2\cos x - x\sin x} = \frac{0}{2-0} = 0$$

Then $\boxed{\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right) = 0}$

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(4) $f(x) = x\ln x$

$f'(x) = \ln x + x\cdot\frac{1}{x} = \ln x + 1$

$f''(x) = \frac{1}{x}$

$f'''(x) = -\frac{1}{x^2}$

$f(1) = 0$

$f'(1) = 1$

$f''(1) = 1$

$f'''(1) = -1$.

Then: $P_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$

i.e. $P_3(x) = 0 + 1\cdot(x-1) + \frac{1}{2}(x-1)^2 + \left(\frac{-1}{3!}\right)(x-1)^3$

i.e. $\boxed{P_3(x) = (x-1) + \frac{1}{2}(x-1)^2 - \frac{1}{6}(x-1)^3}$
