ECUACIONES DIFERENCIALES ORDINARIAS. Miércoles, Marzo 29, 2017

UAM-Azcapotzalco   Examen #3   ANSWER KEY

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(1) The general equation is:

$$m\ddot{x} + b\dot{x} + kx = F(t)$$

Since there is no damping $(b=0)$ and no external forces $(F(t)=0)$ we have.

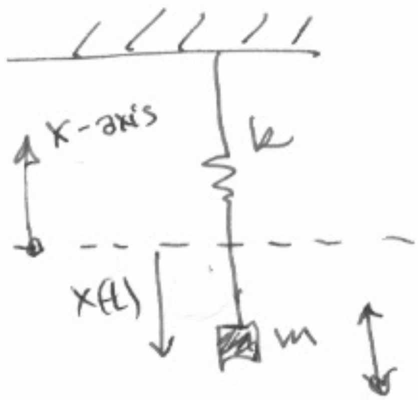
$$m\ddot{x} + kx = 0$$

We have to determine k. Since the gravitational force equilibrates the Hooke's force:

$$F_{\text{Hooke}} = F_{\text{gravitational}} \Rightarrow k\Delta x = mg \Rightarrow k = \frac{mg}{\Delta x}$$

$$\Rightarrow k = \frac{\frac{1}{10}\,\text{kg} \cdot 10\,\text{m/sec}^2}{5/100\ \text{m}} = \frac{100}{5}\ \text{kg/sec}^2 \quad \boxed{k = 20\ \text{N/m}.}$$

Hence, the initial value problem is:



$$\frac{1}{10}\ddot{x} + 20x = 0$$

$$x(0) = 0$$

$$\dot{x}(0) = -\frac{1}{10}\ \text{m/seg}$$

The natural frequency is:

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{20}{1/10}} = \sqrt{200}\ \text{1/sec.}$$

i.e. $\boxed{\omega_0 = \sqrt{2}\,10\ \text{1/sec}} \approx 14.1\ \text{1/seg}$

The solution is: $x(t) = C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)$

$$\left.\begin{array}{l} x(0) = C_1 \\ \dot{x}(0) = \omega_0 C_2 \end{array}\right\} \Rightarrow x(t) = x(0)\cos(\omega_0 t) + \frac{\dot{x}(0)}{\omega_0}\sin(\omega_0 t)$$

$$\Rightarrow \boxed{x(t) = \frac{-1/10}{\sqrt{2}\,10}\sin(\omega_0 t) = \frac{-1}{\sqrt{2}\,100}\sin(\omega_0 t)}$$

The particle returns to the equilibrium position at T, i.e., when

$$\omega_0 T = \pi \Rightarrow T = \frac{\pi}{\omega_0} =$$

(since $\sin(\omega_0 T) = \sin(\pi) = 0$) $\boxed{T = \frac{\pi}{\sqrt{2}\,10}\text{ seg}}$

$\boxed{T \approx 0.22214\text{ seg}}$

(2) The equation is:

$$m\ddot{x} + b\dot{x} + kx = F(t).$$

$m = 5\,kg$ and $F(t) = 10 \sin\left(\frac{t}{2}\right)$. N.

We have to determine b and k.

Again, the gravitational force stretches the string $\frac{1}{10}$ m and it equilibrates Hooke's force: $F_{Hooke} = F_{grav}$. Then

$$k\Delta x = mg \Rightarrow k = \frac{mg}{\Delta x} = \frac{5 \cdot 10\ N}{1/10\ m} = 500\ N/m.$$

Now, the friction force is $F_{frict} = b\,\dot{v}$.

Then: $b = \frac{F_{fric}}{\dot{v}} = \frac{2N}{\frac{4}{100}\ m/seg} = 50\ \frac{N \cdot seg}{m} = 50\ kg/seg$.

Then, the initial value problem reads:

Solve the differential equation:

$$5\ddot{x} + 50\dot{x} + 500x = 10 \sin\left(\frac{t}{2}\right)$$

with initial conditions: $x(0) = 0$ m

$\dot{x}(0) = \frac{8}{100}$ m/seg

As we study in class, the solution is:

$$x(t) = \underbrace{x_u(t)}_{\text{Solution to homogeneous eq'n}} + \underbrace{x_p(t)}_{\text{Particular solution.}}$$

and the homogeneous solution

$x_u(t) \xrightarrow[t \to \infty]{} 0$, independently of being under, over or critically damped,

Then, steady state is $x_p(t)$.

The characteristic eq'n is

$$5r^2 + 50r + 500 = 0$$

$$r^2 + 10r + 100 = 0$$

$$\Rightarrow r_{1,2} = \frac{-10 \pm \sqrt{10^2 - 4\cdot 100}}{2} = \frac{-10 \pm \sqrt{-300}}{2}$$

$$\boxed{r_{1,2} = -5 \pm \frac{\sqrt{3}}{2} 10}$$

Then, the solution to the homogeneous equation has the form:

$$x_h(t) = e^{-5t}\left(C_1 \cos(\sqrt{3}\,5t) + C_2 \sin(\sqrt{3}\,5t)\right) \quad \left(\text{Notice: } x_h(t) \xrightarrow[t\to\infty]{} 0\right)$$

Then, a particular solution has the form:

$$x_p(t) = A\cos\left(\frac{t}{2}\right) + B\sin\left(\frac{t}{2}\right)$$

and it is plausible (Ansatz). The diff. eq'n is

$$\ddot{x} + 10\dot{x} + 100x = 2\sin\left(\frac{t}{2}\right)$$

$$\dot{x}_p = -\frac{1}{2}A\sin\left(\frac{t}{2}\right) + \frac{1}{2}B\cos\left(\frac{t}{2}\right)$$

$$\ddot{x}_p = -\frac{1}{4}\left(A\cos\left(\frac{t}{2}\right) + B\sin\left(\frac{t}{2}\right)\right) = -\frac{1}{4}x_p$$

Hence:

$$-\frac{1}{4}x_p + 10\dot{x}_p + 100x_p = 2\sin\left(\frac{t}{2}\right)$$

$$10\dot{x}_p + \frac{399}{4}x_p = 2\sin\left(\frac{t}{2}\right)$$

$$40\dot{x}_p + 399x_p = 8\sin\left(\frac{t}{2}\right)$$

Then:

$$20B\cos\left(\frac{t}{2}\right) - 20A\sin\left(\frac{t}{2}\right) + 399A\cos\left(\frac{t}{2}\right) + 399B\sin\left(\frac{t}{2}\right) = 8\sin\left(\frac{t}{2}\right)$$

i.e.

$$(20B + 399A)\cos\left(\frac{t}{2}\right) + (-20A + 399B)\sin\left(\frac{t}{2}\right) = 8\sin\left(\frac{t}{2}\right)$$

Hence:
$$20B + 399A = 0 \quad \Rightarrow \quad B = -\frac{399}{20}A$$
$$-20A + 399B = 8 \quad \Rightarrow \quad -20A - \frac{(399)^2}{20}A = 8$$
$$\Rightarrow \quad -\left((20)^2 + (399)^2\right)A = 160$$
$$\Rightarrow \quad \boxed{A = \frac{160}{(20)^2 + (399)^2}} = \frac{160}{159601}$$
$$\boxed{B = \frac{-(399)(80)}{(20)^2 + (399)^2}} = \frac{-31920}{159601}$$
$$\boxed{\begin{aligned} A &\approx 0.001 \\ B &\approx 0.1999 \end{aligned}}$$
Hence, the steady state is:
$$\boxed{x_p(t) = \frac{160}{(20)^2 + (399)^2}\cos\left(\frac{t}{2}\right) - \frac{31920}{(20)^2 + (399)^2}\sin\left(\frac{t}{2}\right)}.$$