UAM-A. Departamento de Ciencias Básicas.
Examen Global de Ecuaciones Diferenciales.
Trimestre 17I. Vespertino

NOTA: El examen global consta de los ejercicios marcados con (*). Si presenta sólo una parte debe resolver TODOS los ejercicios de tal parte. Todos los resultados deben mostrar el procedimiento.

NOMBRE: ANSWER KEY

GRUPO:_

PRIMERA PARTE

1. Resolver

$$y(\ln x)(\ln y)dx + dy = 0$$

2. (* 15 %) Resolver

$$(y + x\cos x)dx + (x\ln x + xe^y)dy = 0$$

3. (* 10 %) Resolver:

$$2y' - \frac{y}{x} = -\frac{x}{y^2}$$

4. (* 15 %) Un cuerpo cuya temperatura es de 90 °C se coloca en el tiempo $t = 0$ en un medio en el que la temperatura se conserva a 20 °C. Si después de 5 minutos el cuerpo se enfrió a 50 °C, determinar la temperatura del cuerpo como función del tiempo. Suponer que la rapidez con qué el cuerpo se enfría es proporcional a la diferencia entre la temperatura del cuerpo y la del medio que lo rodea.

SEGUNDA PARTE

1. (* 10 %) Comprobar que la función $y_1 = x$ es solución de la ecuación diferencial:

$$(1 - x^2)y'' + 2xy' - 2y = 0.$$

Determinar su solución general en el intervalo $(-1, 1)$.

2. (* 15 %) Resolver la ecuación diferencial dada usando el método de coeficientes indeterminados.

$$y'' + 10y' + 25y = 4e^{-5x} + 50x.$$

3. (* 15 %) Resolver la ecuación diferencial:

$$y'' - y = \frac{1}{1 + e^x}.$$

4. Resolver el problema de valores iniciales dado.

$$u'' + 4u = 0$$

$$u(0) = 1, \quad u'(0) = -2.$$

TERCERA PARTE

1. (* 20%) Un cuerpo que pesa 12 lb sujeto al extremo de un resorte lo estira 2 ft. El cuerpo se suelta desde un punto que está 1 ft abajo de la posición de equilibrio, con una velocidad dirigida hacia arriba de 4 ft/s. (a) Determine la ecuación del movimiento, la amplitud y el periodo. (b) ¿En qué instantes pasa el cuerpo por la posición de equilibrio en dirección hacia abajo?

2. Un resorte vertical con constante de 26 lb/pie tiene suspendido un peso de 32 lb. Se aplica una fuerza externa dada por $f(t) = 82\cos 4t, \quad t \geq 0$. Se asume que actúa una fuerza amortiguadora numéricamente igual a 2 veces la velocidad instantánea. Inicialmente el cuerpo está en reposo y 6 ft abajo de la posición de equilibrio. Determinar la posición del peso en cualquier tiempo. ¿Está en resonancia el sistema?

UAM - Azcapotzalco. Martes 4 de abril de 2017

Ecuaciones Diferenciales Ordinarias. Trimestre 17-I

Examen Global

PRIMERA PARTE

(1) Resolver: $y(\ln x)(\ln y)\,dx + dy = 0$

This is a separable equation:

$$y' = -y(\ln x)(\ln y)$$

$$\frac{1}{y\ln y} y' = -\ln x.$$

Integrating with respect to x:

$$\int \frac{1}{y\ln y} y'\,dx = -\int \ln x\,dx$$

First integral Use the change of variables $y = e^u \Rightarrow \frac{dy}{dx} = e^u u'$

$$\int \frac{1}{y\ln y} y'\,dx = \int \frac{1}{e^u u} e^u\,du = \int \frac{1}{u}\,du = \ln u = \ln(\ln y)$$

Second integral: Integration by parts.

$$\int \ln x\,dx = \int (x)' \ln x\,dx = x\ln x - \int x \cdot \frac{1}{x}\,dx = x\ln x - \int dx$$

$$= x\ln x - x = (\ln x - 1)x.$$

Hence:

$$\ln(\ln y) = -(\ln x - 1)x + C_1$$

$$\Rightarrow \ln y = e^{-(\ln x - 1)x} e^{C_1} = C_2 \left(e^{\ln x - 1}\right)^{-x} \quad C_2 = e^{C_1}$$

$$= C_2 (x e^{-1})^{-x} = \cancel{C_2}\, C_2 x^{-x} e^{x} \Rightarrow \boxed{y(x) = e^{C_2 x^{-x} e^{x}}}$$

$= 1 =$

(2) Resolver: $\underbrace{y + x\cos x}_{M(x,y)} + (\underbrace{x\ln x + xe^y}_{N(x,y)})\frac{dy}{dx} = 0$.

Let us check if this is exact equation:

We require $\frac{\partial M}{\partial y}$, $\frac{\partial N}{\partial x}$ to be the same.

$$\left.\begin{aligned}\frac{\partial M}{\partial y} &= 1 \\ \frac{\partial N}{\partial x} &= \ln x + x\frac{1}{x} + e^y \\ &= \ln x + 1 + xe^y\end{aligned}\right\} \text{Not the same. Use an integrating factor:}$$

Multiply by μ: $\quad \mu M + (\mu N)\frac{dy}{dx} = 0$

We require: $\frac{\partial(\mu M)}{\partial y}$, $\frac{\partial}{\partial x}(\mu N)$ to be the same

$$\left.\begin{aligned}\frac{\partial(\mu M)}{\partial y} &= \mu_y M + \mu M_y \\ \frac{\partial(\mu N)}{\partial x} &= \mu_x N + \mu N_x\end{aligned}\right\} \Rightarrow \mu_y M + \mu M_y = \mu_x N + \mu N_x$$

If $\mu_y = 0$: $\quad \mu_x = \mu\left(\frac{M_y - N_x}{N}\right) = \mu\frac{(1-(\ln x + 1 + e^y))}{x\ln x + xe^y}$

$$= \mu\frac{(-(\ln x + e^y))}{x(\ln x + e^y)} = -\frac{1}{x}\mu \Rightarrow \frac{\mu'}{\mu} = -\frac{1}{x}$$

$$\Rightarrow \frac{d}{dx}(\log\mu) = -\frac{d}{dx}(\log x) \Rightarrow \log\mu = -\log x$$

$\log\mu = \log(x^{-1}) \Rightarrow \mu = x^{-1}$ $\quad\boxed{\mu = \frac{1}{x}}$

Multiply eq'n by $\mu = \frac{1}{x}$

$$\underbrace{\left(\frac{y}{x} + \cos x\right)}_{\tilde{M}} + \underbrace{(\ln x + e^y)}_{\tilde{N}}\frac{dy}{dx} = 0 \quad \text{should be exact.}$$

$$\frac{\partial \tilde{M}}{\partial y} = \frac{\partial}{\partial y}\left(\frac{y}{x} + \cos x\right) = \frac{1}{x} + 0$$
$$\frac{\partial \tilde{N}}{\partial} = \frac{\partial}{\partial x}\left(\ln x + e^y\right) = \frac{1}{x} + 0$$
Identical! Exact eq'n.

Hence: $\frac{\partial G}{\partial x} = \frac{y}{x} + \cos x$ ------ I

$\frac{\partial G}{\partial y} = \ln x + e^y$ ------ II

Consider $\frac{\partial G}{\partial y} = \ln x + e^y$. Integrate with respect to y:

$G(x,y) = (\ln x) y + e^y + f(x)$ ---- III

Now, from (III) $\frac{\partial G}{\partial x} = \frac{1}{x} y + f'(x)$.

Compare to: (I); $f'(x) = \cos x \Rightarrow \boxed{f(x) = \sin x}$

Hence; from (III) $G(x,y) = (\ln x) y + e^y + \sin x$.

and the sol'n is: $\boxed{(\ln x) y + e^y + \sin x = C}$

Also, Consider $\frac{\partial G}{\partial x} = \frac{y}{x} + \cos x$ Integrate with respect to x:

$G(x,y) = (\ln x) y + \sin x + h(y)$. ---- IV

From: eq (IV)

$\frac{\partial G}{\partial y} = \ln x + h'(y)$ and compare to eq. (II);

$\Rightarrow h'(y) = e^y \Rightarrow h(y) = e^y$, then, (IV) is:

$G = (\ln x) y + \sin x + e^y$

Same solution.

(3) Solve $2y' - \frac{y}{x} = -xy^{-2}$.

This equation is of the Bernoulli type. Hence, multiply by $\frac{y^2}{2}$:

$$y^2 y' - \frac{1}{2x} y^3 = -\frac{x}{2}$$

Let: $v = y^3 \Rightarrow \frac{dv}{dx} = 3y^2 \frac{dy}{dx} \Rightarrow \frac{1}{3}\frac{dv}{dx} = y^2 y'$

then:
$$\frac{1}{3}\frac{dv}{dx} - \frac{1}{2x} v = -\frac{x}{2} \Rightarrow \boxed{\frac{dv}{dx} - \frac{3}{2x} v = -\frac{3}{2} x}$$

This is a linear, non-homogeneous, constant coefficients diff. eq^n.
Use integrating factors:

$$\mu \frac{dv}{dx} - \frac{3}{2x} \mu v = -\frac{3\mu}{2} x.$$

Hence: $\frac{d\mu}{dx} = -\frac{3}{2x}\mu \Rightarrow \frac{1}{\mu}\frac{d\mu}{dx} = -\frac{3}{2x} \Rightarrow \frac{d}{dx}(\log \mu) = -\frac{3}{2}\frac{d}{dx}(\log x)$

$\Rightarrow \log \mu = -\frac{3}{2} \log x \Rightarrow \boxed{\mu(x) = x^{-3/2}}$.

(Const integration $= 0$

Hence:

$$\mu \frac{dv}{dx} - \frac{3}{2x}\mu v = -\frac{3}{2}\mu x \Rightarrow \mu\frac{dv}{dx} + \frac{d\mu}{dx} v = -\frac{3}{2}\mu x \Rightarrow \frac{d}{dx}(\mu v) = -\frac{3}{2}\mu x.$$

$$\Rightarrow \frac{d}{dx}\left(x^{-3/2} v\right) = -\frac{3}{2} x^{-3/2} \cdot x = -\frac{3}{2} x^{-1/2} \Rightarrow x^{-3/2} v = -\frac{3}{2}\int x^{-1/2} dx$$

$$\Rightarrow x^{-3/2} v = -3x^{1/2} + C \Rightarrow v(x) = -3x^2 + Cx^{3/2}$$

Since $y^3(x) = v(x) \Rightarrow \boxed{y(x) = \sqrt[3]{-3x^2 + Cx^{3/2}}}$

(1) Newton's cooling law:

$$\frac{dT}{dt} = -k(T - T_A). \quad (k > 0)$$

$T_A = 20°C$, and $T(0) = 90°C$ is the initial condition

If $[t] = min$, $T(5) = 50°C$, find k and $T(t)$

The solution to Newton's cooling law is.

$$T(t) = (T(0) - T_A)\,e^{-kt} + T_A$$

Now, at $t = 5_{min}$, $T = 20°C$:

$$T(5) = (T(0) - T_A)\,e^{-5k} + T_A$$

Solving for k:

$$\frac{T(5) - T_A}{T(0) - T_A} = e^{-5k} \Rightarrow e^{5k} = \frac{T(0) - T_A}{T(5) - T_A} \quad \boxed{k = \frac{1}{5}\,\mathrm{Log}\left(\frac{T(0) - T_A}{T(5) - T_A}\right)}$$

i.e. $\boxed{k = \frac{1}{5}\,\mathrm{Log}\,\frac{7}{3}}$ i.e. $k \sim 0.169$ 1/min

Then: $T(t) = (90 - 20)\,e^{-kt} + 20°C$ $\boxed{T = 70\,e^{-\left(\frac{1}{5}\log\frac{7}{3}\right)t} + 20°C}$

$$\Rightarrow \boxed{T(t) = 70\left(\frac{7}{3}\right)^{-t/5} + 20\ °C}$$

SEGUNDA PARTE.

(1) The equation is: $(1-x^2)y'' + 2xy' - 2y = 0$

The function $y_1(x) = x$, has derivatives $y_1'(x) = 1$, $y_1''(x) = 0$.

Hence: $(1-x^2)\cdot 0 + 2x\cdot 1 - 2(x) = 2x - 2x = 0$, eq'n holds.

Now, by reduction of order:

$$y_2(x) = A(x)\, y_1(x) = x\,A(x).$$

$$y_2'(x) = A' y_1 + A y_1' = xA' + A$$

$$y_2''(x) = A'' y_1 + 2A' y_1' + A y_1'' = xA'' + 2A'$$

Substituting into eq'n:

$$(1-x^2)(xA'' + 2A') + 2x(xA' + A) - 2xA = 0.$$

Grouping in factors of A'', A', A:

$$x(1-x^2)A'' + [2(1-x^2) + 2x\cdot x]A' + 2xA - 2xA = 0.$$

$$x(1-x^2)A'' + [2 - 2x^2 + 2x^2]A' = 0$$

$$x(1-x^2)A'' + 2A' = 0$$

Let $B(x) \equiv A'(x)$. Then, $\dfrac{B'}{B} = \dfrac{-2}{x(1-x^2)}$

By partial fractions.

$$\frac{-2}{x(1-x^2)} = \frac{-2}{x(1-x)(1+x)} = \frac{\alpha}{x} + \frac{\beta}{1-x} + \frac{\gamma}{1+x}.$$

$$= \frac{\alpha(1-x)(1+x) + \beta x(1+x) + \gamma x(1-x)}{x(1-x)(1+x)}$$

Hence: $-2 = \alpha(1-x)(1+x) + \beta(x)(1+x) + \gamma x(1-x)$.

Choose: $x=0$: $-2 = \alpha\cdot 1\cdot 1 \Rightarrow \boxed{\alpha = -2}$

$x=+1$: $-2 = \beta\cdot 1(2) \Rightarrow \boxed{\beta = -1}$

$x=-1$: $-2 = \gamma(-1)(2) \Rightarrow \boxed{\gamma = 1}$

Hence:

$$\frac{B'}{B} = -\frac{2}{x} - \frac{1}{1-x} + \frac{1}{x+1} \Rightarrow \frac{B'}{B} = -\frac{2}{x} + \frac{1}{x-1} + \frac{1}{x+1}$$

$$\Rightarrow \frac{d}{dx}(\operatorname{Log} B) = -\frac{2}{x} + \frac{1}{x-1} + \frac{1}{x+1} \Rightarrow \operatorname{Log}|B| = -2\log|x| + \log|x-1| + \log|x+1| + \log C$$

(C = const.)

$$\Rightarrow \log|B| = \operatorname{Log}|x^{-2}(x-1)(x+1)C| \Rightarrow \underline{B(x) = C\frac{(x-1)(x+1)}{x^2}}$$

Now $A'(x) = C\frac{(x^2-1)}{x^2} = C\left(1 - \frac{1}{x^2}\right) \Rightarrow A(x) = C\left(x + \frac{1}{x}\right) + D$.

D = const.

$$\Rightarrow y_2(x) = xA(x) = C(x^2+1) + Dx.$$

$$C_1 y_1(x) + C_2 y_2(x) = C_1 x + C_2\left(C(x^2+1) + Dx\right)$$

$$= (C_1 + C_2 D)x + C_2 C(x^2+1).$$

Choose: $C=1$, $D=0$:

$$\boxed{y(x) = C_1 x + C_2(x^2+1)}$$

② The following equation

$$y_h'' + 10y_h' + 25y_h = 0$$

is:
1) homogeneous
2) Constant coefficient
3) linear,

then, it has solutions of the form.

$$y_h = e^{rx}.$$

Then $r^2 + 10r + 25 = 0 \Rightarrow (r+5)^2 = 0 \Rightarrow r_1 = r_2 = -5$ repeated roots

Solution: $y_h(x) = (C_1 + C_2 x)e^{-5x}$.

Propose particular solutions:

$$\left.\begin{aligned} y_p(x) &= Ae^{-5x}, \text{ not judicious} \\ y_p(x) &= Axe^{-5x}, \text{ not judicious} \end{aligned}\right\} \text{Repeats } y_h(x)$$

$$y_p(x) = Ax^2 e^{-5x}, \text{ Judicious!}.$$

$$y_p'(x) = 2Axe^{-5x} - 5Ax^2e^{-5x} = Ax(2-5x)e^{-5x}$$

$$y_p''(x) = A(2e^{-5x} - 2\cdot 2\cdot 5xe^{-5x} + 25x^2e^{-5x})$$

$$= A(2 - 20x + 25x^2)e^{-5x}$$

Substitute onto the diff. eq'n:

$$A(2-20x+25x^2)e^{-5x} + 10Ax(2-5x)e^{-5x} + 25Ax^2e^{-5x} = 4e^{-5x}$$

Hence:

$$Ae^{-5x}(2 - \cancel{20x} + \cancel{25x^2} + \cancel{20x} - \cancel{50x^2} + \cancel{25x^2}) = 4e^{-5x}$$

$$A(2) = 4 \Rightarrow \boxed{A = 2}$$

The other particular solution has the following form:

$$y_p(x) = \alpha x + \beta, \text{ it is judicious!}$$

$$y_p'(x) = \alpha$$

$$y_p''(x) = 0.$$

Then:

$$0 + 10\alpha + 25(\alpha x + \beta) = 50x.$$

$$(25\alpha)x + (10\alpha + 25\beta) = 50x.$$

then:

$$\left.\begin{array}{l} 25\alpha = 50 \\ 10\alpha + 25\beta = 0 \end{array}\right\} \Rightarrow \begin{array}{l} \boxed{\alpha = 2} \\ 2\alpha + 5\beta = 0 \Rightarrow \beta = -\frac{2\alpha}{5} \end{array}$$

Then, the general solution is: $\boxed{\beta = -\frac{4}{5}}$

$$\boxed{y(x) = (C_1 x + C_2)e^{-5x} + 2x^2 e^{-5x} + 2x - \frac{4}{5}}$$

③ Use variation of parameters. We require first to solve the homogeneous equation:

$$y_h'' - y_h = 0 \Rightarrow y_h(x) = C_1 e^{x} + C_2 e^{-x}.$$

Let $y_1(x) = e^{x}$ and $y_2(x) = e^{-x}$.

The particular solution has the form:

$$y_p(x) = A(x)\, y_1(x) + B(x)\, y_2(x)$$

where:

$$A(x) = -\int \frac{y_2(x)\, g(x)}{a(x)\, W[y_1, y_2](x)}\, dx$$

$$B(x) = \int \frac{y_1(x)\, g(x)}{a(x)\, W[y_1, y_2](x)}\, dx.$$

Here, the Wronskian is:

$$W[y_1, y_2] = \det\begin{pmatrix} e^{x} & e^{-x} \\ e^{x} & -e^{-x} \end{pmatrix} = -1 - 1 = -2$$

and $a(x) = 1$, $g(x) = \frac{1}{1+e^{x}}$. Hence, we have:

$$A(x) = -\int \frac{e^{-x}}{1(-2)} \frac{1}{1+e^{x}}\, dx = \frac{1}{2} \int \frac{e^{-x}}{1+e^{x}}\, dx$$

Let $u = e^{-x}$, $du = -e^{-x}\, dx$

$$A(x) = \frac{1}{2} \int \frac{-du}{1+u^{-1}} = -\frac{1}{2} \int \frac{u}{1+u}\, du = -\frac{1}{2} \int \frac{1+u-1}{1+u}\, du$$

$$= -\frac{1}{2} \int 1 - \frac{1}{1+u}\, du = -\frac{1}{2} \int du + \frac{1}{2} \int \frac{du}{1+u} = -\frac{u}{2} + \frac{1}{2} \operatorname{Log}|1+u|$$

$$= -\frac{e^{-x}}{2} + \frac{1}{2} \operatorname{Log}|1+e^{-x}| = \frac{1}{2}\left(\operatorname{Log}|1+e^{-x}| - e^{-x}\right)$$

Now:

$$B(x) = \int \frac{y_1(x)\, g(x)}{a(x)\, W[y_1, y_2](x)}\, dx = \int \frac{e^x}{(1)(-2)} \frac{1}{1+e^x}\, dx =$$

$$= -\frac{1}{2} \int \frac{e^x}{1+e^x}\, dx \underset{\uparrow}{=} -\frac{1}{2} \int \frac{du}{u} = -\frac{1}{2} \operatorname{Log}|u| = -\frac{1}{2} \operatorname{Log}|1+e^x|$$

$u = 1+e^x$

$du = e^x dx$

Hence, the general solution is:

$$\boxed{y(x) = C_1 e^x + C_2 e^{-x} + \frac{1}{2}\left(\operatorname{Log}(1+e^{-x}) - e^{-x}\right) e^x - \frac{1}{2}\left(\operatorname{Log}(1+e^x)\right) e^{-x}}$$

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(4) The solution to $u'' + 4u = 0$

i.e. $u'' = -4u$

is $\boxed{u(t) = C_1 \cos(2t) + C_2 \sin(2t)}$

TERCERA PARTE.

(1) $w = 12\,lb$. We have to determine Hooke's constant.

$$w = k\Delta x \Rightarrow k = \frac{w}{\Delta x} = \frac{12\,lb}{2\,ft} = 6\,lb/ft.$$

The mass of the body, m, satisfies:

$$mg = w$$

$$m \cdot 32\,ft/sec^2 = 12\,lb.$$

$$\Rightarrow m = \frac{12}{32} = \frac{6}{16} = \frac{3}{8}\,\frac{lb \cdot sec^2}{ft}.$$

The equation is:

$$m\ddot{y} + ky = 0 \Rightarrow \frac{3}{8}\ddot{y} + 6y = 0 \qquad \boxed{\ddot{y} + 16y = 0}$$

$$\boxed{y(t) = y(0)\cos(4t) + \frac{\dot{y}(0)}{w_0}\sin(4t)}, \text{ where } w_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{6}{3/8}} = \sqrt{16}$$

i.e. $\boxed{w_0 = 4 \cdot 1/sec.}$

Since $y(0) = -1\,ft$, $\dot{y}(0) = 4\,ft/sec$.

then: $y(t) = -\cos(4t) + \frac{4}{4}\sin(4t) \Rightarrow \boxed{y(t) = -\cos(4t) + \sin(4t)\ ft.}$

Also: $\boxed{y(t) = A\cos(4t - \varphi_0)}$

where the amplitude $A = \sqrt{(-1)^2 + 1^2} \Rightarrow \boxed{A = \sqrt{2}}$

Now: $y(t) = A\cos(4t)\cos\varphi_0 + A\sin(4t)\sin\varphi_0$

$$\Rightarrow \left.\begin{aligned} A\cos\varphi_0 &= -1 \\ A\sin\varphi_0 &= 1 \end{aligned}\right\} \Rightarrow \varphi_0 \in \text{Quadrant II}, \quad \varphi_0 \in \left(\frac{\pi}{2}, \pi\right).$$

$$\tan\varphi_0 = \frac{A\cos\varphi_0}{A\sin\varphi_0} = -1 \Rightarrow \varphi_0 = \operatorname{Arctan}(-1) + \pi = -\frac{\pi}{4} + \pi$$

$$\boxed{\varphi_0 = \frac{3}{4}\pi}$$

Then: $y(t) = \sqrt{2}\cos\left(4t - \frac{3\pi}{4}\right)$.

The period P is such that:

$$\cos\left(4(t+P) - \frac{3\pi}{4}\right) = \cos\left(4t - \frac{3\pi}{4}\right)$$

Then: $4t + \underline{\underline{4P}} - \frac{3\pi}{4} = \cos 4t + \underline{\underline{2\pi}} - \frac{3\pi}{4}$.

$$\Rightarrow \quad 4P = 2\pi \quad \Rightarrow \quad \boxed{P = \frac{\pi}{2}}$$

is the period

(b) Now, $\cos\left(4t_n - \frac{3\pi}{4}\right) = 0$

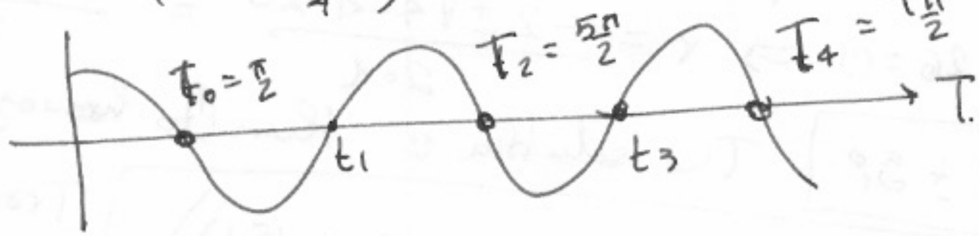
$4t_n - \frac{3\pi}{4} = (2n+1)\frac{\pi}{2}$, $n = 0, 1, 2, 3, \dots$

$4t_n = n\pi + \frac{\pi}{2} + \frac{3\pi}{4} = n\pi + \frac{5}{4}\pi$

$$\boxed{t_n = n\frac{\pi}{4} + \frac{5\pi}{16}}$$

Here, $y(t_n) = 0$

$\cos\left(4t - \frac{3}{4}\pi\right) = \cos T$



$y(t_0) = 0 = y(t_2) = y(t_4)$

$\dot{y}(t_0) = \dot{y}(t_2) = \dot{y}(t_4) < 0$ $\Big\} \Rightarrow n$ is even.

$n = 2m$, $m = 0, 1, 2$

Then: $t_m = 2m\frac{\pi}{4} + 5\frac{\pi}{16}$

$t_n = n\frac{\pi}{4} + \frac{5\pi}{16} = t_m$

$$\Rightarrow \quad \boxed{t_m = m\frac{\pi}{2} + \frac{5\pi}{16}}$$

(2) The equation is: $m\ddot{y} + b\dot{y} + ky = F(t)$.

Since $w = 32\,lb$ and $mg = w \Rightarrow m = \frac{w}{g}$

$$m = \frac{32\,lb}{32\,ft/sec^2} = 1 \cdot \frac{lb \cdot sec^2}{ft}.$$

$k = 26\,\frac{lb}{ft}$ is given. Then, the natural frequency:

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{26}{1}} = \sqrt{26}\,/sec.$$

Now:

$$F(t) = F_0 \cos(\omega t) = 82\cos(4t).$$

The external frequency: is: $\omega = 4\,/sec.$

The friction force is $F_f = 2v$ $\Rightarrow b = \frac{F_f}{v} = 2\,\frac{lb \cdot sec.}{ft}$

Also $F_f = bv$

the equation of motion is:

$$\boxed{\ddot{y} + 2\dot{y} + 26y = 82\cos(4t).}$$

The characteristic eq'n is:

$$r^2 + 2r + 26 = 0 \Rightarrow r = \frac{-2 \pm \sqrt{4 - 4\cdot 26}}{2\cdot 1} = \frac{-2 \pm 2\sqrt{-25}}{2}$$

$\boxed{r_{1,2} = -1 \pm 5i}$ The solution is then: (to homogeneous eq)

$$\boxed{y_h(t) = e^{-t}\left(C_1\cos(5t) + C_2\sin(5t)\right)}$$ Transient solution.

A particular solution (steady state) is given as: (Undetermined Coefficients)

$$y_p(t) = A\cos 4t + B\sin 4t$$

$$\dot{y}_p(t) = -4A\sin 4t + 4B\cos 4t$$

$$\ddot{y}_p = -16y_p = -16(A\cos 4t + B\sin 4t).$$

Substitute:

$$-16y_p + 2(-4A\sin t + 4B\cos 4t) + 26y_p = 82\cos 4t.$$

$$10y_p + (-8A\sin t + 8B\cos 4t) = 82\cos 4t$$

$$(10A\cos 4t + 10B\sin 4t) + (-8A\sin 4t + 8B\cos 4t) = 82\cos 4t.$$

$$(10A + 8B)\cos 4t + (10B - 8A)\sin 4t = 82\cos 4t$$

$$\Rightarrow \left.\begin{matrix} 10A + 8B = 82 \\ -8A + 10B = 0 \end{matrix}\right\} \Rightarrow \begin{matrix} 5A + 4B = 41 \\ -4A + 5B = 0 \end{matrix}$$

$$\begin{pmatrix} 5 & 4 \\ -4 & 5 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 41 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{41}\begin{pmatrix} 5 & -4 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} 41 \\ 0 \end{pmatrix}$$

$$A = \frac{1}{41}\cdot(5\cdot 41 + 0) = 5 \qquad \boxed{\begin{matrix} A = 5 \\ B = 4 \end{matrix}}$$

$$B = \frac{1}{4}(4\cdot 41 + 0) = 4$$

Then the solution, i.e. the position of the particle at any time is:

$$\boxed{y(t) = (C_1\cos 5t + C_2\sin 5t)e^{-t} + 5\cos(4t) + 4\sin(4t)}$$

Now $\dot{y}(t) = 5(-C_1\sin 5t + C_2\cos 5t)e^{-t}$ $- 20\sin 4t$

$$- (C_1\cos 5t + C_2\sin 5t)e^{-t}$$

$$- 20\sin 4t + 16\cos 4t.$$

Now. $y(0) = C_1 + 5 = -6\,ft \Longrightarrow \boxed{C_1 = -11\,ft}$

$\dot{y}(0) = 5C_2 - C_1 + 16 = 0\,ft \Longrightarrow 5C_2 + 11 + 16 = 0$

$$\Rightarrow \boxed{C_2 = -\frac{27}{5}\,ft}$$

$$\boxed{y(t) = \left(-11\cos(5t) - \frac{27}{5}\sin(5t)\right)e^{-t} + 5\cos(4t) + 4\sin(4t)}$$

is the solution to the I.V.P.

The quasi-frequency is $\boxed{\mu = 5/\text{sec.}}$

Resonance: If ω_0 is the natural frequency and $F(t) = F_0\cos(\omega t)$ is the external force with frequency ω,

the amplitude as a function of ω is given by

$$A(\omega) = \frac{F_0}{\sqrt{m^2(\omega^2-\omega_0^2)^2 + b^2\omega}}, \quad \begin{cases} F_0 = 82 \\ \omega_0 = \sqrt{26} \\ b = 2 \end{cases} \quad k = 2$$

$A(\omega_{\text{max}})$ is maximum at $\omega^2 = \omega_1^2 = \omega_0^2\left(1 - \frac{b^2}{2km}\right)$

$$\omega_1^2 = 26\left(1 - \frac{4}{2\cdot 26\cdot 1}\right) = 26\left(1 - \frac{2}{26}\right)$$

$$= 26\left(\frac{26-2}{26}\right) = 24 \Rightarrow \boxed{\omega_1 = \sqrt{24}}$$

Since $\omega = 4 \neq \omega_1 = \sqrt{24}$, then there is no resonance