Quiz #3 A Ecuaciones Diferenciales Ordinarias | Viernes, Febrero 10, 2017

(1) The equation

$$2xy + (y^2 - 3x^2)y' = 0.$$

is not separable, neither linear, nor exact.

$$M = 2xy$$
$$N = y^2 - 3x^2$$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= 2x \\ \frac{\partial N}{\partial x} &= -6x \end{aligned}\right\} \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \quad \Rightarrow \underline{\underline{no}} \text{ es } \underline{\text{exacta}}$$

$\Rightarrow$ factor integrante.

Multiply by μ, then, to be exact are requir

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N) \Rightarrow \mu_y M + \mu M_y = \mu_x N + \mu N_x.$$

If $\mu_y = 0 \Rightarrow \mu = \mu(x)$ only $\Rightarrow \mu\left(\frac{M_y - N_x}{N}\right) = \mu_x$

$$\Rightarrow \mu_x = \mu\left(\frac{2x + 6x}{y^2 - 3x^2}\right) = \frac{8x}{y^2 - 3x^2}\mu,$$

but the factor depends on both, x and y

If $\mu_x = 0 \Rightarrow \mu = \mu(y)$ only $\Rightarrow \mu_y = \mu\frac{(N_x - M_y)}{M}.$

$$\Rightarrow \mu_y = \mu\frac{-6x - 2x}{2xy} \Rightarrow \mu_y = -\frac{8x}{2xy}\mu$$

$$\Rightarrow \mu_y = -\frac{4}{y}\mu \Rightarrow \frac{\mu_y}{\mu} = -\frac{4}{y}$$

$$\Rightarrow \frac{d}{dy}\left(\log|\mu|\right) = -4\frac{d}{dy}\left(\log|y|\right) \Rightarrow \log|\mu| = -4\log|y|$$

$$\Rightarrow \mu = y^{-4}$$

Then: $(2xy)y^{-4} + (y^2-3x^2)y^{-4}y' = 0$ is exact!

$$\frac{\partial G}{\partial x} = (2xy)y^{-4} = \frac{2x}{y^3} \quad \text{--- (I)}$$

$$\frac{\partial G}{\partial y} = y^{-2} - 3x^2y^{-4} \quad \text{--- (II)}$$

From eq (I) $\frac{\partial G}{\partial x} = \frac{2x}{y^3} \Rightarrow G(x,y) = \frac{x^2}{y^3} + f(y)$

$\Rightarrow \frac{\partial G}{\partial y} = -3\frac{x^2}{y^4} + f'(y)$. Com

Compare with (II) $\Rightarrow f'(y) = y^{-2} \Rightarrow f(y) = -y^{-1} + C_1$

$$\Rightarrow \quad G(x,y) = \frac{x^2}{y^3} - \frac{1}{y} + C_1$$

$G(x,y) = C \Rightarrow \boxed{\frac{x^2}{y^3} - \frac{1}{y} = C}$ is an implicit solution

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(2) $\frac{dy}{dx} + \frac{y}{x} = x^2y^2$.

It is a Bernoulli equation, with $\alpha = 1-n = 1-2 = -1$

Then, using the transformation.

$$v(x) = y^\alpha = y^{-1}.$$

$$\frac{dv}{dx} = -y^{-2}\frac{dy}{dx} = -y^{-2}\left(-\frac{y}{x} + x^2y^2\right) = +\frac{y^{-1}}{x} - x^2 = \frac{v}{x} - x^2$$

$\Rightarrow \frac{dv}{dx} - \frac{v}{x} = -x^2$, which is linear and can be solved by the integrating factor method

The integrating factor $\mu(x) = e^{\int -\frac{1}{x} dx} = e^{-\log|x|}$

$$\Rightarrow \quad \mu(x) = \frac{1}{x}.$$

And $\int \mu(x) g(x)\, dx = \int \frac{1}{x}(-x^2)\, dx = -\int x\, dx = -\frac{x^2}{2}.$

Hence:

$$\frac{1}{\mu(x)} \int \mu(x) g(x)\, dx = \left(\frac{1}{1/x}\right)\left(-\frac{x^2}{2}\right) = -\frac{x^3}{2}.$$

$$\Rightarrow \quad v(x) = \frac{C}{\mu(x)} + \frac{1}{\mu(x)} \int \mu(x) g(x)\, dx = \frac{C}{1/x} - \frac{x^3}{2} = \frac{Cx - x^3}{2}$$

$$\Rightarrow \quad y(x) = v^{-1}(x) \Rightarrow \boxed{y(x) = \frac{2}{Cx - x^3}}$$

Quiz #3 - B

① Solve the differential equation:

$$(x^4 - x + y) - xy' = 0$$

It is not exact: $M = x^4 - x + y$

$N = -x$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= +1 \\ \frac{\partial N}{\partial x} &= -1 \end{aligned}\right\} \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.$$

Multiply by μ: $\mu M + \mu N y' = 0$.

Exact if: $\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N) \Rightarrow \mu_y M + \mu M_y = \mu_x N + \mu N_x$.

If $\mu_y = 0 \Rightarrow \mu = \mu(x) \Rightarrow \mu \frac{(M_y - N_x)}{N} = \mu_x \Rightarrow \mu_x = \frac{1-(-1)}{-x}\mu$

$\Rightarrow \mu_x = -\frac{2}{x}\mu \Rightarrow \frac{\mu_x}{\mu} = -\frac{2}{x} \Rightarrow \log|\mu| = -2\log|x|$

$\Rightarrow \mu = x^{-2}$.

Hence: $x^{-2}(x^4 - x + y) - x^{-2} x y' = 0$ is exact.

i.e. $(x^2 - x^{-1} + yx^{-2}) - \frac{y'}{x} = 0$. is exact.

Then $\left.\begin{aligned} \frac{\partial G}{\partial x} &= x^2 - x^{-1} + yx^{-2} \\ \frac{\partial G}{\partial y} &= -\frac{1}{x} \end{aligned}\right\} \begin{aligned} &\Rightarrow G(x,y) = \frac{x^3}{3} - \log|x| - yx^{-1} + f(y) \\ &\Rightarrow \frac{\partial G}{\partial y} = -x^{-1} + f'(y) \end{aligned}$

$\Rightarrow \frac{\partial G}{\partial y}$ $f'(y) = 0 \Rightarrow f'(y) = \text{const.} \Rightarrow G(x,y) = \frac{x^3}{3} - \log|x| - \frac{y}{x} + C$

$$\frac{x^3}{3} - \log|x| - \frac{y}{x} = C$$

$$\Rightarrow \boxed{y(x) = \frac{x^4}{3} - x\log|x| + Cx}$$

4 =

(2) Resuelva la ec. diferencial.

$$\frac{dy}{dx} - 2\frac{y}{x} = -x^2y^2$$

$n = 2$

It is a Bernoulli Dif. Equation: $\alpha = 1 - n = -1$.

$$\Rightarrow \qquad v(x) = y^{-1}(x)$$

$$\frac{dv}{dx} = -y^{-2}\frac{dy}{dx} = -y^{-2}\left(2\frac{y}{x} - x^2y^2\right) = -2\frac{y^{-1}}{x} + x^2 = -2\frac{v}{x} + x^2$$

$$\Rightarrow \quad \frac{dv}{dx} + \frac{2}{x}v = +x^2$$

By integrating factors: $\mu(x) = e^{\int \frac{2}{x}dx} = e^{2\log|x|} = x^2$.

Hence: $\int \mu(x)\, g(x)\, dx = \int x^2(+x^2)\,dx = +\frac{x^5}{5}$

Then, the sol'n is:

$$v(x) = \frac{1}{\mu(x)}\int \mu(x)\,g(x)\,dx + \frac{C}{\mu(x)} = \frac{1}{x^2}\left(+\frac{x^5}{5}\right) + \frac{C}{x^2}$$

$$v(x) = Cx^{-2} - \frac{x^3}{5} = \frac{5C + x^5}{5x^2}$$

Hence $\boxed{y(x) = \frac{5x^2}{C - x^5}}$