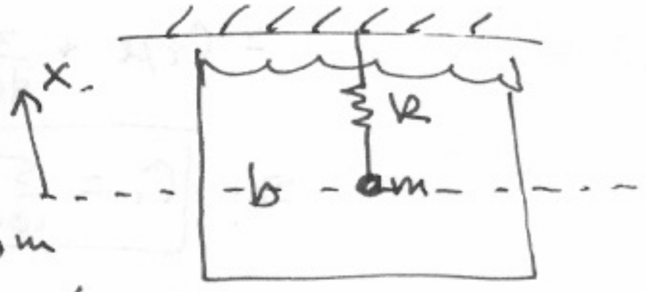
UAM-Azcapotzalco. Friday, Marzo 24, 2017

Quiz # 9 Solution Key

(1) We have the equation of motion:

$$m\ddot{x} + b\dot{x} + kx = 0$$



$m = 2\,kg$

The initial conditions $x(0) = -5/100\ m$

$\dot{x}(0) = 1/10\ m/sec$

We have to determine k and b:

$$F = k\Delta x \Rightarrow k = \frac{F}{\Delta x} = \frac{3\ N}{1/10\ m} = 30\ N/m.$$

$$F = bv \Rightarrow b = \frac{F}{v} = \frac{3\ N}{5\ m/seg} = \frac{3}{5} \cdot \frac{N \cdot seg}{m}$$

Then:

$$2\ddot{x} + \frac{3}{5}\dot{x} + 30x = 0$$

i.e.

$$10\ddot{x} + 3\dot{x} + 150x = 0$$

The characteristic equation $10r^2 + 3r + 150 = 0.$

The roots: $r_{1,2} = \dfrac{-3 \pm \sqrt{9 - 4 \cdot 1500}}{2 \cdot 10} = \dfrac{-3 \pm \sqrt{-5991}}{20}$

$$\simeq -\frac{3}{20} \pm 3.87i = \lambda \pm i\mu$$

The solution is:

$$\boxed{x(t) = \left(C_1 \cos(\mu t) + C_2 \sin(\mu t)\right) e^{-\frac{3}{20}t}.}$$

~~At~~ $\dot{x}(t) = \left(-C_1 \sin(\mu t) + C_2 \cos(\mu t)\right) \mu e^{-\frac{3}{20}t} +$

$$+ \left(-\frac{3}{20}\right) x(t)$$

At $t=0$:

$$-5/100 = x(0) = C_1$$

$$-1/10 = \dot{x}(0) = C_2\mu + \left(\frac{-3}{20}\right)x(0) = C_2\mu - \frac{3}{20}\left(\frac{-5}{100}\right)$$

$$= C_2\mu + \frac{3}{400} \Rightarrow C_2\mu = -\frac{1}{10} - \frac{3}{400} = -\frac{43}{400}$$

$$\Rightarrow \boxed{C_1 = \frac{5}{100}}, \quad \boxed{C_2 = -\frac{43}{400\mu}} \approx 0.0277.$$

Hence:

$$\boxed{x(t) = \left(\frac{5}{100}\cos\mu t - \frac{43}{400\mu}\sin\mu t\right)e^{-\frac{3}{20}t}}$$

The quasi frequency: $\boxed{\mu = \frac{\sqrt{5991}}{20}}$

$$w_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{30}{2}} = \sqrt{15} \quad \boxed{w_0 = \sqrt{15}} \approx 3.9\ldots/\text{sec}$$

$\approx 3.87/\text{sec}$.

Ratio: $\boxed{\frac{\mu}{w_0} = \frac{1}{20}\sqrt{\frac{5991}{15}} \approx 0.999249}$

(2) The circuit has the following characteristics:

$C = 10^{-5}$ F

$R = 3\times10^{2}\ \Omega$

$\mathcal{L} = \frac{2}{10}$ H.

Initial Conditions:

$Q(0) = 10^{-6}$ C

$\dot{Q}(0) = 0$ A.

Find Q at any time. The diff. eq'n is:

$$\mathcal{L}\ddot{Q} + R\dot{Q} + \frac{1}{C}Q = 0.$$

Then,

$$\frac{2}{10}\ddot{Q} + 300\dot{Q} + 10^{5}Q = 0.$$

The roots of

$$\frac{2}{10}r^2 + 300r + 10^5 = 0$$

i.e.

$$2r^2 + 3000r + 10^6 = 0$$

are:

$$r = \frac{-3000 \pm \sqrt{9\times10^6 - 4(2)10^6}}{2(2)}$$

i.e.

$$r = \frac{-3000 \pm \sqrt{10^6}}{4} = \frac{-3000 \pm 10^3}{4} = \begin{cases} -500\ \text{1/sec} = r_1 \\ -1000\ \text{1/sec} = r_2 \end{cases}$$

The solution is:

$$Q(t) = C_1e^{r_1t} + C_2e^{r_2t} = C_1e^{-1000t} + C_2e^{-2000t}.$$

Now $\dot{Q}(t) = r_1C_1e^{r_1t} + r_2C_2e^{r_2t}$.

At $t=0$: $Q(0) = C_1 + C_2 = 10^{-6}$

$\dot{Q}(0) = r_1C_1 + r_2C_2 = 0 \Rightarrow C_2 = -\frac{r_1C_1}{r_2}$

$$\Rightarrow C_1 - \frac{r_1}{r_2}C_1 = 10^{-6} \Rightarrow \left(1-\frac{r_1}{r_2}\right)C_1 = 10^{-6} \Rightarrow \left(1-\frac{1}{2}\right)C_1 = 10^{-6}$$

$\frac{1}{2} C_1 = 10^{-6}$ $\boxed{C_1 = 2\times 10^{-6} \text{ Coulombs}}$

$\Rightarrow$ $\boxed{C_2 = -\frac{r_1}{r_2} C_1 = -\frac{1}{2}(2\times 10^{-6}) = \boxed{-10^{-6}\, C}}$

$\Rightarrow$ $\boxed{Q(t) = (2e^{-500t} - e^{-1000t})\times 10^{-6} \text{ Coulombs}}$

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(3) (a) We have the system:

$$\ddot{x} + \gamma\dot{x} + x = 0.$$

The roots to the characteristic equation:

$$r^2 + \gamma r + 1 = 0$$

are:

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4}}{2}$$

To be underdamped, we require: $\gamma^2 - 4 < 0$, i.e.:

$$0 < \gamma < 2.$$

$$r = \frac{-\gamma \pm i\sqrt{4-\gamma^2}}{2}$$

The quasi-frequency is then $\mu = \frac{\sqrt{4-\gamma^2}}{2}$

The quasiperiod is then $P_{quasi} = \frac{2\pi}{\mu} = \frac{4\pi}{\sqrt{4-\gamma^2}}$

For the undamped motion, the natural frequency is: $\omega_0 = \sqrt{\frac{k}{m}}$

Here. $\ddot{x} + x = 0$, i.e. $\begin{matrix} k=1 \\ m=1 \end{matrix} \Rightarrow \omega_0 = 1.$

The natural period becomes: $P = \frac{2\pi}{\omega_0} = 2\pi.$

It is required: $P_{quasi} = (1.5)P$

Hence: $$\frac{4\pi}{\sqrt{4-\gamma^2}} = \frac{3}{2}\cdot 2\pi$$

i.e.: $$\frac{4}{3} = \sqrt{4-\gamma^2} \Rightarrow \frac{16}{9} = 4-\gamma^2 \Rightarrow \gamma^2 = 4-\frac{16}{9}$$

$$\gamma^2 = \frac{36-16}{9} = \frac{20}{9} = \frac{4\cdot 5}{9} \Rightarrow \boxed{\gamma = \frac{2}{3}\sqrt{5}} \boxed{\approx 1.49}$$

(b) To have a critically damped system, in the equation
$$r = \frac{-\gamma \pm \sqrt{\gamma^2-4}}{2}$$
we require a repeated root, i.e.:
$$\sqrt{\gamma^2-4} = 0$$
$\Rightarrow \gamma^2 = 4 \Rightarrow \underline{\gamma = \pm 2}$. But since the damping is always positive: $\boxed{\gamma = 2}$
