Ecuaciones Diferenciales Ordinarias Winter Quarter 2017.
Examen #1 - A. Miercoles 15 de febrero de 2017
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ANSWER KEY

(1) $\frac{dy}{dx} = \frac{xy+y^2}{x^2}$.

$$f(\alpha x, \alpha y) = \frac{(\alpha x)(\alpha y) + (\alpha y)^2}{(\alpha x)^2} = \frac{\alpha^2(xy+y^2)}{\alpha^2 x^2} = \frac{xy+y^2}{x^2} = f(x,y)$$

f is an homogeneous function of order cero.

We can make the change of variables: $v(x) = \frac{y(x)}{x}$

Hence $y(x) = xv(x)$, $y' = xv' + v$

Hence, the diff. eq:

$$\frac{dy}{dx} = \frac{xy}{x^2} + \frac{y^2}{x^2} = \frac{y}{x} + \left(\frac{y}{x}\right)^2 \text{ becomes } xv' + v = v + v^2$$

Then: $\frac{v'}{v^2} = \frac{1}{x} \Rightarrow \frac{d}{dx}\left(\frac{-1}{v}\right) = \frac{d}{dx}(\log x)$

$$\Rightarrow \frac{-1}{v} = \log|x| + C \Rightarrow -v = \frac{1}{\log|x| + C}$$

Since $v(x) = \frac{y(x)}{x}$: $-\frac{y(x)}{x} = \frac{1}{\log|x| + C}$

$$\Rightarrow \boxed{y(x) = \frac{-x}{\log|x| + C}} \text{ or } \boxed{y(x) = \frac{x}{C - \log|x|}}$$

Remark. This equation can also be solved by the integrating factor. It is also a Bernoulli equation.

(2) $-x^2 \frac{dy}{dx} + (y^2 + 2xy) = 0$

$M = y^2 + 2xy \qquad \frac{\partial M}{\partial y} = 2y + 2x$

$N = -x^2 \qquad \frac{\partial N}{\partial x} = -2x$

It is <u>not</u> <u>exact</u>.

Let $\mu(x,y)$ be an integrating factor.

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N) \Rightarrow \mu_y M + \mu M_y = \mu_x N + \mu N_x .$$

$$\text{If } \mu_y = 0 \Rightarrow \mu\underline{\left(\frac{M_y - N_x}{N}\right)} = \mu_x \Rightarrow \mu_x = \mu\left(\frac{4x+2y}{-x^2}\right)$$

It does not work, since depends on x, y.

$$\text{If } \mu_x = 0 \Rightarrow \mu_y = \left(\frac{N_x - M_y}{M}\right)\mu \Rightarrow \mu_y = \frac{-2x-2y-2x}{y^2+2xy}$$

$$\Rightarrow \mu_y = \frac{-2(2x+y)}{y(y+2x)} \Rightarrow \mu_y = -\frac{2}{y}\mu$$ depends only on y.

$$\Rightarrow \frac{\mu_y}{\mu} = -\frac{2}{y} \Rightarrow \operatorname{Log}\mu = -2\log y \Rightarrow \boxed{\mu(y) = y^{-2}}$$

Then, the diff. eq'n becomes

$$(y^2 + 2xy)\, y^{-2} - x^2 y^{-2}\frac{dy}{dx} = 0$$ is <u>exact</u>!

i.e. $$\left(1 + \frac{2x}{y}\right) - \frac{x^2}{y^2}\frac{dy}{dx} = 0 .$$

Then:

$$\frac{\partial G}{\partial x} = 1 + \frac{2x}{y} \Rightarrow G(x,y) = x + \frac{x^2}{y} + h(y)$$

Now: $$\frac{\partial G}{\partial y} = -\frac{x^2}{y^2} + h'(y)$$

Comparing with $\frac{\partial G}{\partial y} = -\frac{x^2}{y^2}$, implies: $h'(y)=0 \Rightarrow h(y) = \text{const}$

thus: $G(x,y) = x + \frac{x^2}{y} + \text{const}$

Therefore, the solution (implicit!): $G(x,y) = C$

becomes

$$\boxed{x + \frac{x^2}{y^2} = C}$$

C=constant.

(3) $\frac{dy}{dx} + \frac{y}{x} = x^3 y^2$ is a Bernoulli equation:

$$y^{-2}\frac{dy}{dx} + \frac{1}{x} y^{-1} = x^3$$

Let $v(x) = y^{-1}(x) \Rightarrow \frac{dv}{dx} = -y^{-2}\frac{dy}{dx}$. thus:

$$-\frac{dv}{dx} + \frac{1}{x} v = x^3 \quad \therefore \text{ Linear, } 1^{st} \text{ order, non-homogeneous eq'n.}$$

$$\frac{dv}{dx} - \frac{1}{x} v = -x^3.$$

The integrating factor is: $\mu(x) = e^{-\int \frac{1}{x}dx} = e^{-\log|x|} = x^{-1}$.

Now:
$$\int \mu(x) g(x)\, dx = \int x^{-1}(-x^3)\, dx = -\int x^2\, dx = -\frac{x^3}{3}.$$

thus:
$$v(x) = \frac{C_1}{\mu} + \frac{1}{\mu(x)}\int \mu(x) g(x)\, dx = \frac{C_1}{x^{-1}} + \frac{1}{x^{-1}}\left(\frac{-x^3}{3}\right)$$

$$= C_1 x - \frac{x^4}{3}, \quad C_1 - \text{constant}.$$

Since $y(x) = \frac{1}{v(x)} \Rightarrow \boxed{y(x) = \frac{3}{C_2 x - x^4}}$

C_2 - new constant

4) We need Newton's cooling law:

$$\frac{dT}{dt} = -k(T - T_a).$$

with initial condition $T(0) = 95°C$. and $T_a = 21°C$.

The solution is $T(t) = Ce^{-kt} + T_a$.

At $t=0$, $T(0) = C + T_a \Rightarrow C = T(0) - T_a = 74°C$.

$$T(t) = (T(0) - T_a)e^{-kt} + T_a \quad \boxed{T(t) = 74e^{-kt} + 21\ °C}$$

It remains to determine k.

When $t = 5_{min}$, $T(5) = 80°C$.

$$(T(0) - T_a)e^{-5k} + T_a = T(5)$$

$$\Rightarrow e^{-5k} = \frac{T(5) - T_a}{T(0) - T_a} \Rightarrow \boxed{k = \frac{1}{5}\log\left(\frac{T(0) - T_a}{T(5) - T_a}\right)}$$

Then: $\boxed{k = \frac{1}{5}\log\left(\frac{95-21}{80-21}\right)} = \boxed{\frac{1}{5}\log\frac{74}{59}\ \frac{1}{min}} \approx 0.045\ \frac{1}{min}$.

We want t_1 when $T(t) = 50°C$.

$$(T(0) - T_a)e^{-kt} + T_a = 50°C : \text{Solve for } t:$$

$$e^{-kt} = \frac{50 - T_a}{T(0) - T_a} \Rightarrow \boxed{t = \frac{1}{k}\log\left(\frac{T(0) - T_a}{50 - T_a}\right) = \frac{1}{k}\log\left(\frac{74}{29}\right)}$$

$$\Rightarrow \boxed{t = 5\frac{\log(74/29)}{\log(74/59)}\ mins} \approx 20.81\ mins$$