ODE's Exam #1-B UAM-A. Winter Quarter 2017

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ANSWER KEY

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(1) $\frac{dy}{dx} = -\left(\frac{x^2+y^2}{2xy}\right)$

$$f(\alpha x,\alpha y) = -\left(\frac{(\alpha x)^2+(\alpha y)^2}{2(\alpha x)(\alpha y)}\right) = -\left(\frac{\alpha^2(x^2+y^2)}{\alpha^2(2xy)}\right) = -\frac{(x^2+y^2)}{(2xy)} = f(x,y).$$

f is an homogeneous function of order 0. Then, define

$v(x) = \frac{y(x)}{x} \Rightarrow y(x) = x v(x) \Rightarrow y' = v + xv'$.

The diff. eqn: $\frac{dy}{dx} = -\frac{x}{2y} - \frac{1}{2}\frac{y}{x}$.

because: $v + xv' = -\frac{1}{2v} - \frac{1}{2}v \Rightarrow xv' = -\frac{1}{2v} - \frac{3}{2}v$

i.e. $xv' = \frac{-1-3v^2}{2v}$ is separable: $\frac{2v}{1+3v^2}\frac{dv}{dx} = -\frac{1}{2x}$.

$\Rightarrow \frac{1}{6}\log(1+3v^2) = -\frac{1}{2}\log x + C_1 \Rightarrow (1+3v^2)^{1/6} = C_2 x^{-1/2}$

(with $C_1 = \log C_2$)

$\Rightarrow 1+3v^2 = C_3 x^{-3}$ (with $C_3 = C_2^6$).

$\Rightarrow v(x) = \pm\sqrt{\frac{Cx^{-3}-1}{3}} \Rightarrow \boxed{y(x) = \pm x\sqrt{\frac{Cx^{-3}-1}{3}}}$

Remark: It can also be solved by integrating factors, or by a Bernoulli equation.

② $$\underbrace{(2xy^3+1)}_{M} + \underbrace{(3x^2y^2-y^{-1})}_{N}\frac{dy}{dx} = 0$$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= 6xy^2 \\ \frac{\partial N}{\partial x} &= 6xy^2 \end{aligned}\right\} \quad \text{It is } \underline{\underline{\text{exact}}}!$$

$$\left.\begin{aligned} \frac{\partial G}{\partial x} &= 2xy^3+1 \\ \frac{\partial G}{\partial y} &= 3x^2y^2-y^{-1} \end{aligned}\right\} \Rightarrow \quad G(x,y) = x^2y^3 + x + h(y)$$

$$\Downarrow$$

$$\frac{\partial G}{\partial y} = 3x^2y^2 + h'(y)$$

Comparing. $h'(y) = -y^{-1} \Rightarrow h(y) = -\operatorname{Log} y + C_1$

$$\Rightarrow \quad G(x,y) = x^2y^3 + x - \operatorname{Log} y + C_1$$

Then the solution $G(x,y) = C_1$

is $$\boxed{x^2y^3 + x - \operatorname{Log} y = C_1}$$

(3) $\frac{dy}{dx} - y = e^{2x} y^2$ is a Bernoulli eq'n:

$$y^{-2}\frac{dy}{dx} - y^{-1} = e^{2x}.$$

Let $v(x) = y^{-1}$; $\Rightarrow \frac{dv}{dy} = -y^{-2}\frac{dy}{dx}$.

then::

$$-\frac{dv}{dy} - v = e^{2x} \Rightarrow \frac{dv}{dy} + v = -e^{2x}.$$

Linear eq'n. Non-homogeneous.

Integrating factor:

$$\mu(x) = e^{\int p(x)\,dx} = e^{\int dx} = e^{x}.$$

Also:

$$\int \mu(x) g(x)\,dx = \int e^{x}(-e^{2x})\,dx = -\int e^{3x}\,dx = -\frac{1}{3}e^{3x}$$

$$\Rightarrow v(x) = \frac{c}{\mu} + \frac{1}{\mu}\int \mu(x) g(x)\,dx = \frac{C_1}{e^{x}} + \frac{1}{e^{x}}\left(-\frac{1}{3}e^{3x}\right)$$

$$\Rightarrow v(x) = C_1 e^{-x} - \frac{1}{3}e^{2x}$$

$$\boxed{y(x) = \frac{1}{v(x)} = \frac{3}{C_2 e^{-x} - e^{2x}}}$$

(with $C_2 = 3C_1$ new constant)

(4) We use Newton's cooling law

$$\frac{dT}{dt} = -k(T - T_a) \quad , \quad \text{with} \quad \begin{array}{l} T(0) = 2^\circ C \\ T_a = 19^\circ C. \end{array}$$

The solution is: $T(t) = Ce^{-kt} + T_a$.

At $t=0$: $T(0) = C + T_a \Rightarrow C = T(0) - T_a$.

$$\Rightarrow T(t) = (T(0) - T_a)e^{-kt} + T_a \Rightarrow \boxed{T(t) = -17e^{-kt} + 19^\circ C}$$

When $t = 3$ min, $T(3) = 4^\circ C$; hence:

$$(T(0) - T_a)e^{-3k} + T_a = T(3) \Rightarrow e^{-3k} = \frac{T(3) - T_a}{T(0) - T_a}$$

$$\Rightarrow \boxed{k = \frac{1}{3}\log\left(\frac{T_a - T(0)}{T_a - T(3)}\right) = \frac{1}{3}\log\frac{17}{15}\frac{1}{\text{min}}} \approx 0.042\frac{1}{\text{min}}$$

Then: $T(t) = -17e^{-kt} + 19^\circ C$.

When: $T(t) = 10^\circ C$? Solve. $-17e^{-kt} + 19 = 10^\circ C$.

i.e. $(T(0) - T_a)e^{-kt} + T_a = T_{final} \Rightarrow e^{-kt} = \frac{T_f - T_a}{T(0) - T_a}$

$$\Rightarrow \boxed{t = \frac{1}{k}\log\left(\frac{T_a - T(0)}{T_a - T_f}\right) = \frac{1}{k}\log\left(\frac{17}{9}\right) \text{mins}}$$

i.e. $\boxed{t = \frac{3}{\log(17/15)} \cdot \log\left(\frac{17}{9}\right) \text{mins}}$

i.e. $\boxed{t \approx 15.14 \text{ min} \approx 15' : 09''}$