UNIVERSIDAD AUTÓNOMA METROPOLITANA
DEPARTAMENTO DE CIENCIAS BÁSICAS

EXAMEN GLOBAL CÁLCULO INTEGRAL
Trimestre 17-O. TURNO VESPERTINO

Alumno:
Grupo: ANSWER KEY

El examen global consta de los 8 problemas marcados con ♣. Quienes presenten **sólo** una parte del examen deberán resolver **todos** los problemas correspondientes a esa parte. **Todos los ejercicios deberán mostrar el procedimiento correspondiente en su solución.**

PRIMERA PARTE

♣(1)(5%) Calcular la derivada de la función:

$$F(t) = \int_{\tan^2(\sqrt{t})}^{0} e^{-x^2}\, dx$$

Calcular las integrales:

♣(2)(15%) $\displaystyle\int_1^9 \frac{dx}{(1+\sqrt{x})^2}$ **♣(3)(15%)** $\displaystyle\int \frac{(\ln x)^2}{x^2}\, dx$ (4) $\displaystyle\int_2^5 \left(t - \frac{1}{t}\right)^2 dt$

SEGUNDA PARTE

Resolver

(5) $\displaystyle\int \tan^3(2x)\sec^3(2x)\, dx$ **♣(6)(15%)** $\displaystyle\int \frac{x^3}{\sqrt{2-4x^2}}\, dx$ **♣(7)(15%)** $\displaystyle\int \frac{-2x^2+x+2}{x^3+x}\, dx$

♣ (8)(10%) Calcular el valor de la siguiente integral impropia: $\displaystyle\int_1^{+\infty} 7xe^{-x}\, dx$

TERCERA PARTE

♣ (9)(10%) Calcular el área de la región limitada por la gráfica de $y = 2^{1-x}$ y la recta $y = 2x - 1$, para $x \in [0, 1]$.

♣ (10)(15%) Calcular el volumen del sólido de revolución obtenido al rotar alrededor del eje X la región del primer cuadrante limitada por las gráfica de $y = \cos x$, $y = \operatorname{sen} x$ y la recta $x = 0$.

(11) Calcular la longitud de la curva $y = \sqrt{4-x^2}$ para $x \in [-1, 1]$.

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CÁLCULO INTEGRAL Examen Global | Jueves, Diciembre 14, 2017.

ANSWER KEY

PRIMERA PARTE.

(1) $$F(t) = \int_{\tan^2(\sqrt{t})}^{0} e^{-x^2}\,dx = -\int_{0}^{\tan^2(\sqrt{t})} e^{-x^2}\,dx$$

Then
$$\frac{dF}{dt} = -e^{-(\tan^2\sqrt{t})^2}\frac{d}{dt}\tan^2(\sqrt{t}),$$ by the Fundamental Theorem of Calculus and the Chain rule.

$$= -e^{-\tan^4\sqrt{t}}\, 2\tan(\sqrt{t})\sec^2(\sqrt{t})\cdot\frac{1}{2\sqrt{t}},$$ by the Chain rule again

$$\boxed{\frac{dF}{dt} = -\frac{e^{-\tan^4(\sqrt{t})}\tan(\sqrt{t})\sec^2(\sqrt{t})}{\sqrt{t}}}$$

(2) $$\int_1^9 \frac{dx}{(1+\sqrt{x})^2} = \int_1^3 \frac{1}{(1+y)^2}\frac{dx}{dy}\,dy = \int_1^3 \frac{2y}{(1+y)^2}\,dy.$$

Change of variable: $y = \sqrt{x}$.

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} \Rightarrow \frac{dx}{dy} = 2\sqrt{x} = 2y.$$

$x = 1 \Rightarrow y = \sqrt{1} = 1$

By parts: $x = 9 \Rightarrow y = \sqrt{9} = 3$

$$= \int_1^3 \frac{d}{dy}\left[\frac{-1}{1+y}\right]2y = -\frac{1}{1+y}2y\Big|_1^3 + \int_1^3\frac{2}{1+y}\,dy = \left(-\frac{2y}{1+y} + 2\operatorname{Log}(1+y)\right)\Big|_1^3$$

$$= -2\left(\frac{3}{4}-\frac{1}{2}\right) + 2\left(\operatorname{Log}(4) - \operatorname{Log}2\right) = -2\left(\frac{1}{4}\right) + 2\operatorname{Log}\left(\frac{4}{2}\right)$$

$$= -\frac{1}{2} + 2\operatorname{Log}2. \qquad \boxed{\int_1^4 \frac{dx}{(1+\sqrt{x})^2} = 2\log 2 - \frac{1}{2}}$$

(3) $\int \frac{(\operatorname{Log} x)^2}{x^2}\,dx = \int \frac{y^2}{(e^{y})^2}\frac{dx}{dy}\,dy = \int \frac{y^2}{(e^y)^2} e^y\,dy$

Let $\begin{cases} y = \operatorname{Log} x \\ e^y = x \end{cases}$; $\frac{dy}{dx} = \frac{1}{x}$; $\frac{dx}{dy} = x = e^y$

i.e.

By parts.

$= \int \frac{y^2}{e^y}\,dy = \int y^2 e^{-y}\,dy = -y^2 e^{-y} + \int 2y e^{-y}\,dy$

By parts

$= -y^2 e^{-y} - 2y e^{-y} + \int 2e^{-y}\,dy = -y^2 e^{-y} - 2ye^{-y} - 2e^{-y} + C$

$= -(y^2 + 2y + 2)e^{-y} + C.$

i.e.

$$\boxed{\int \frac{(\operatorname{Log} x)^2}{x^2}\,dx = -\frac{(\operatorname{Log}^2 x + 2\operatorname{Log} x + 2)}{x} + C}$$

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(4) $\int_2^5 \left(t - \frac{1}{t}\right)^2 dt = \int_2^5 t^2 - 2 + \frac{1}{t^2}\,dt = \left(\frac{1}{3}t^3 - 2t - \frac{1}{t}\right)\Big|_2^5$

$= \frac{1}{3}(5^3 - 2^3) - 2(5-2) - \left(\frac{1}{5} - \frac{1}{2}\right) = \frac{1}{3}(125 - 8) - 6 - \left(\frac{2-5}{10}\right) =$

$= \frac{117}{3} - 6 + \frac{3}{10} = \frac{1170 - 180 + 9}{30} = \frac{999}{30} = \boxed{\frac{333}{10}}$

$$\boxed{\int_2^5 \left(t - \frac{1}{t}\right)^2 dt = \frac{333}{10}}$$

SEGUNDA PARTE.

(5) $\displaystyle\int \tan^3(2x)\sec^3(2x)\,dx \overset{y=2x}{=} \frac{1}{2}\int \tan^3(y)\sec^3(y)\,dy$

$\displaystyle = \frac{1}{2}\int \tan^2 y \sec^2 y \,(\sec(y)\tan(y))\,dy = \frac{1}{2}\int (\sec^2 y - 1)\sec^2 y \frac{d}{dy}(\sec y)\,dy$

$z = \sec y$:

$\displaystyle = \frac{1}{2}\int (z^2-1)\, z^2\,dz = \frac{1}{2}\int z^4 - z^2 = \frac{1}{2}\left(\frac{z^5}{5} - \frac{1}{3}z^3\right) + C$

$\displaystyle = \frac{1}{2}\left(\frac{1}{5}\sec^5(2x) - \frac{1}{3}\sec^3(2x)\right) + C$

i.e.

$$\boxed{\int \tan^3(2x)\sec^3(2x)\,dx = \frac{\sec^3(2x)}{2}\left(\frac{\sec^2(2x)}{5} - \frac{1}{3}\right) + C}$$

(6) $\displaystyle\int \frac{x^3}{\sqrt{2-4x^2}}\,dx = \frac{1}{\sqrt{2}}\int \frac{x^3}{\sqrt{1-2x^2}}\,dx$

Let $\sqrt{2}\,x = \cos\theta$.

Then $\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-2x^2}$

and: $x = \frac{1}{\sqrt{2}}\cos\theta \Rightarrow \frac{dx}{d\theta} = \frac{1}{\sqrt{2}}(-\sin\theta) = -\frac{1}{\sqrt{2}}\sqrt{1-2x^2}$

$\displaystyle = \frac{1}{\sqrt{2}}\int \frac{\left(\frac{1}{\sqrt{2}}\cos\theta\right)^3}{\sin\theta}\frac{dx}{d\theta}\,d\theta = \frac{1}{\sqrt{2}}\frac{1}{(\sqrt{2})^3}\int \frac{\cos^3\theta}{\sin\theta}\left(-\frac{1}{\sqrt{2}}\sin\theta\right)d\theta$

$\displaystyle = -\frac{1}{(\sqrt{2})^5}\int \cos^3\theta\,d\theta = -\frac{1}{(\sqrt{2})^4\sqrt{2}}\int \cos^2\theta\cos\theta\,d\theta$

$\displaystyle = -\frac{1}{4\sqrt{2}}\int (1-\sin^2\theta)\cos\theta\,d\theta \underset{y=\sin\theta}{=} -\frac{1}{4\sqrt{2}}\int (1-y^2)\,dy$

$$= -\frac{1}{4\sqrt{2}}\left(y - \frac{1}{3}y^3\right) = \frac{1}{4\sqrt{2}}\left(\frac{y^3}{3} - y\right) + C_1$$

$$= \frac{y}{4\sqrt{2}}\left(\frac{y^2}{3} - 1\right) + C_1 = \frac{\sin\theta}{4\sqrt{2}}\left(\frac{\sin^2\theta}{3} - 1\right) + C$$

$$= \frac{\sqrt{1-2x^2}}{4\sqrt{2}}\left(\frac{1-2x^2}{3} - 1\right) + C = \frac{\sqrt{1-2x^2}}{4\sqrt{2}}\left(\frac{1}{3} - 1 - \frac{2}{3}x^2\right) + C$$

$$= \frac{\sqrt{1-2x^2}}{4\sqrt{2}}\left(-\frac{2}{3} - \frac{2}{3}x^2\right) + C$$

$$\boxed{\int \frac{x^3}{\sqrt{2-4x^4}}\,dx = -\frac{1}{6\sqrt{2}}\sqrt{1-2x^2}\,(1+x^2) + C_1}$$

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(7) $$\int \frac{-2x^2 + x + 2}{x^3 + x}\,dx = \int \frac{-2x^2 + x + 2}{x(x^2+1)}\,dx$$

Partial fractions:

$$\frac{-2x^2 + x + 2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx + C}{x^2+1} = \frac{Ax^2 + A + Bx^2 + Cx}{x(x^2+1)}$$

$$= \frac{(A+B)x^2 + Cx + A}{x(x^2+1)} \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l} A + B = -2 \\ \boxed{\begin{array}{l} C = 1 \\ A = 2 \end{array}} \end{array}\right\} \Rightarrow 2 + B = -2 \Rightarrow \boxed{B = -4}$$

Hence:

$$\int \frac{-2x^2 + x + 2}{x^3 + x}\,dx = \int \left(\frac{2}{x} + \frac{-4x + 1}{x^2+1}\right)dx = \int\left(\frac{2}{x} - 2\left(\frac{2x}{x^2+1}\right) + \frac{1}{x^2+1}\right)dx$$

$$\boxed{= 2\,\mathrm{Log}|x| - 2\log(1+x^2) + \mathrm{Arctan}\,x + C_1}$$

(8) We have to compute the limit:

$$\int_1^\infty 7xe^{-x}\,dx = \lim_{M\to\infty} 7\int_1^M x e^{-x}\,dx.$$

So we need:

$$\int xe^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} + C.$$

$$= -(x+1)e^{-x} + C'.$$

then,

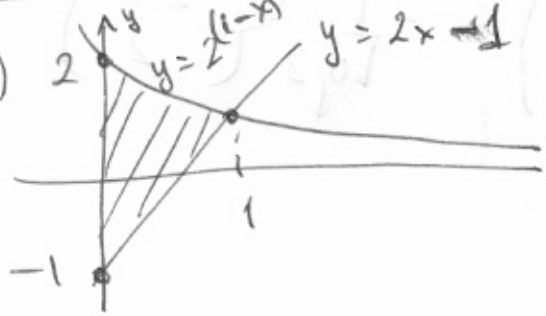
$$\int_1^\infty 7xe^{-x}\,dx = -7\lim_{M\to\infty}(1+x)e^{-x}\Big|_1^M$$

$$= -7\lim_{x\to\infty}\left((1+M)e^{-M} - 2e^{-1}\right)$$

Since $\lim_{M\to\infty} e^{-M} = 0$ and $\lim_{M\to\infty} Me^{-M} = 0$, we have:

$$= -7(-2e^{-1}) \Rightarrow \boxed{\int_1^\infty 7xe^{-x}\,dx = 14e^{-1}}$$

(9)



Notice that, at $x=1$:

$$2^{1-x} = 2x-1 \text{ holds:}$$

$$2^0 = 2-1$$

$$1 = 1 \checkmark$$

$$\text{Area} = \int_0^1 2^{1-x}\,dx - \int_0^1 (2x-1)\,dx = 2\int_0^1 2^{-x}\,dx - (x^2-x)\Big|_0^1$$

$$= 2\int_0^1 2^{-x}\,dx - [0-0] = 2\int_0^1 2^{-x}\,dx$$

It remains to compute: $\int_0^1 2^{-x}\,dx$

Let $y = \log(2^x)$ i.e. $e^y = 2^x$

and $y = x\log 2.$ $\left\{\begin{array}{l} x=1 \Rightarrow y=\log \\ x=0 \Rightarrow y=0 \end{array}\right.$

Then: $x = \frac{1}{\log 2}y \Rightarrow \frac{dx}{dy} = \frac{1}{\log 2}$

Therefore:

$$\int_0^1 2^{-x}\,dx = \int_0^{\log 2} e^{-y}\frac{dx}{dy}\,dy = \int_0^{\log 2}\frac{e^{-y}}{\log 2}\,dy =$$

$$= -\frac{e^{-y}}{\log 2}\Big|_0^{\log 2} = -\frac{1}{\log 2}\left(e^{-\log 2}-1\right)$$

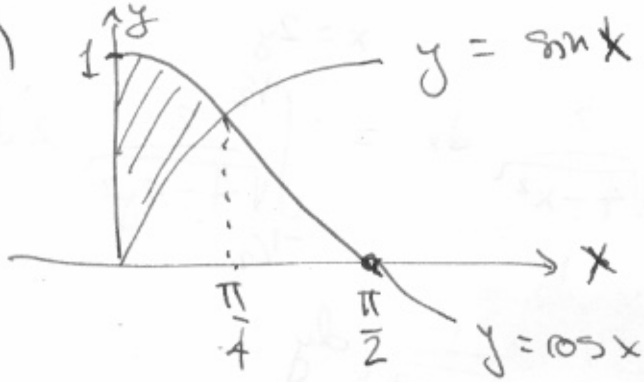
$$= -\frac{1}{\log 2}\left(2^{-1}-1\right) = -\frac{1}{\log 2}\left(-\frac{1}{2}\right)$$

$$= \frac{1}{2\log 2}.$$

Then:

$$\text{Area} = 2\int_0^1 2^{-x}\,dx = 2\left(\frac{1}{2\log 2}\right) = \frac{1}{\log 2}$$

$$\boxed{\text{Area} = \frac{1}{\log 2}}$$

(10) 

$\sin x = \cos x = \frac{1}{\sqrt{2}}$

at $x = \frac{\pi}{4}$.

Then: $\text{Volume} = V_1 - V_2 = \pi \int_0^{\pi/4} \cos^2 x \, dx - \pi \int_0^{\pi/4} \sin^2 x \, dx$

$= \pi \int_0^{\pi/4} (\cos^2 x - \sin^2 x) \, dx = \pi \int_0^{\pi/4} \cos(2x) \, dx$

$= \pi \frac{1}{2} \sin 2x \Big|_0^{\pi/4} = \frac{\pi}{2}\left(\sin\frac{\pi}{2} - \sin 0\right) = \frac{\pi}{2}\cdot(1-0)$

$\boxed{\text{Volume} = \frac{\pi}{2}}$

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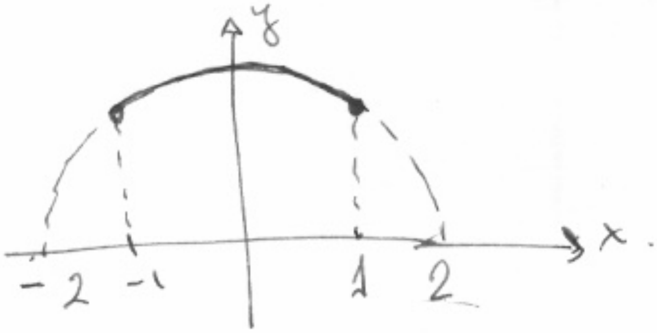
(11) $y = \sqrt{4-x^2}$. Then $x^2 + y^2 = 4$. Then, we have a portion of a circle:



We need to compute:

$$\int_{-1}^{1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

Hence. $\frac{dy}{dx} = \frac{-x}{\sqrt{4-x^2}}$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2}{4-x^2} = \frac{4-x^2+x^2}{4-x^2} = \frac{4}{4-x^2}$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{2}{\sqrt{4-x^2}}$$

Hence:

$$\ell = \int_{-1}^{1} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \int_{-1}^{1} \frac{2}{\sqrt{4-x^2}}\, dx \overset{x=2y}{=} \int_{-1/2}^{1/2} \frac{2}{\sqrt{4-4y^2}}\, 2\,dy$$

$$= \int_{-1/2}^{1/2} \frac{4}{2\sqrt{1-y^2}}\, dy = 2\int_{-1/2}^{1/2} \frac{1}{\sqrt{1-y^2}}\, dy$$

$$= 4\int_0^{1/2} \frac{1}{\sqrt{1-y^2}}\, dy,$$ since the integrand is an <u>even</u> function

$$= 4 \operatorname{Arcsin} y \Big|_0^{1/2} = 4\left(\operatorname{Arcsin}\left(\tfrac{1}{2}\right) - \operatorname{Arcsin} 0\right)$$

$$= 4\left(\frac{\pi}{6} - 0\right)$$

$$\boxed{\text{Length} = \frac{2\pi}{3}}$$