UAM - Azcapotzalco. CÁLCULO INTEGRAL (Viernes 5 de enero 2018)

Examen de Recuperación 17-O.

(1). Por las propiedades de la integral.

$$g(x) = \int_{x^2}^{x^{1/3}} \tan^5(t)\,dt = \int_{x^2}^{0} \tan^5(t)\,dt + \int_{0}^{x^{1/3}} \tan^5(t)\,dt.$$

$$= -\int_0^{x^2} \tan^5(t)\,dt + \int_0^{x^{1/3}} \tan^5(t)\,dt.$$

Sean $y(x) = x^2$; $Y(x) = x^{1/3}$.

$$g(x) = -\int_0^{y(x)} \tan^5(t)\,dt + \int_0^{Y(x)} \tan^5(t)\,dt.$$

Entonces. por las propiedad de la suma de funciones y su derivada:

$$\frac{dg}{dx} = -\frac{d}{dx}\left(\int_0^{y(x)} \tan^5(t)\,dt\right) + \frac{d}{dx}\left(\int_0^{Y(x)} \tan^5(t)\,dt\right)$$

Regla de la Cadena:

$$= -\frac{d}{dy}\left(\int_0^{y} \tan^5(t)\,dt\right)\frac{dy}{dx} + \frac{d}{dY}\left(\int_0^{Y} \tan^5(t)\,dt\right)\frac{dY}{dx}$$

Por el Teorema Fundamental del Cálculo:

$$= -\tan^5(y)\frac{dy}{dx} + \tan^5(Y)\frac{dY}{dx}$$

Así

$$\frac{dg}{dx} = -\tan^5(x^2)\,2x + \tan^5(x^{1/3})\left(\frac{1}{3}x^{-2/3}\right)$$

i.e.

$$\boxed{\frac{dg}{dx} = -2x\tan^5(x^2) + \frac{\tan^5(x^{1/3})}{3x^{2/3}}}$$

② (a) $\displaystyle\int_1^e \frac{(\ln x)^7}{x}\,dx.$

Take $y = \ln x$. Then: $\dfrac{dy}{dx} = \dfrac{1}{x}$.

$x = e \longrightarrow y = \ln e = 1$

$x = 1 \longrightarrow y = \ln 1 = 0$

Thence:

$$\int_1^e (\ln x)^7 \frac{1}{x}\,dx = \int_1^e (y(x))^7 \frac{dy}{dx}\,dx = \int_0^1 y^7\,dy = \frac{1}{8} y^8 \Big|_0^1$$

i.e. $\boxed{\displaystyle\int_1^e \frac{(\ln x)^7}{x}\,dx = \frac{1}{8}}$

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② (b) $\displaystyle\int (x+4)\frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx.$

METHOD 1

Changing variables: $z = \sqrt{x}$; $\dfrac{dz}{dx} = \dfrac{1}{2\sqrt{x}}$

$$\int (x+4)\,e^{\sqrt{x}}\left(\frac{1}{\sqrt{x}}\right)dx = \int (x+4)\,e^{\sqrt{x}}\left(2\,\frac{dz}{dx}\right)dx.$$

$$= \int (z^2+4)\,e^z \cdot 2\,dz = 2\int (z^2+4)\,e^z\,dz$$

$$= 2\int z^2 e^z\,dz + 8\int e^z\,dz.$$

By parts

$$= 2\left(z^2e^z - \int 2z e^z\,dz\right) + 8e^z.$$

$$= 2\left(z^2 e^z - 2\int z e^z dz\right) + 8e^z$$

$$\underset{\substack{\uparrow\\ \text{By parts}}}{=} 2\left[z^2e^z - 2\left(ze^z - \int e^z\,dz\right)\right] + 8e^z$$

$$= 2\left[z^2e^z - 2\left(ze^z - e^z\right)\right] + 8e^z + C_1.$$

$$= 2\left[z^2e^z - 2ze^z + 2e^z\right] + 8e^z + C_1.$$

$$= 2z^2e^z - 4ze^z + 4e^z + 8e^z + C.$$

$$= 2e^z\left(z^2 - 2z + 6\right) + C_1$$

i.e.

$$\boxed{\int (x+4)\frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2e^{\sqrt{x}}\left(x - 2\sqrt{x} + 6\right) + C_1}$$

<u>METHOD 2.</u> $\int (x+4)\frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx.$

Notice that: $\frac{d}{dx}\left(e^{\sqrt{x}}\right) = \frac{1}{2\sqrt{x}}e^{\sqrt{x}}$, i.e. $\frac{d}{dx}\left(2e^{\sqrt{x}}\right) = \frac{e^{\sqrt{x}}}{\sqrt{x}}$.

Hence:

$$\int (x+4)\frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int (x+4)\frac{d}{dx}\left(e^{\sqrt{x}}\right)dx \underset{\substack{\uparrow\\ \text{By parts}}}{=} 2\left[(x+4)e^{\sqrt{x}} - \int \frac{d}{dx}(x+4)e^{\sqrt{x}}\,dx\right]$$

$$= 2(x+4)e^{\sqrt{x}} - 2\int e^{\sqrt{x}}\,dx$$

Now: $\int e^{\sqrt{x}}\,dx \underset{\sqrt{x}=y}{=} \int e^y\,2y\,dy \underset{\text{By parts}}{=} 2\left(ye^y - e^y\right) + C'$

i.e. $\int e^{\sqrt{x}}\,dx = 2(\sqrt{x}-1)e^{\sqrt{x}} + C.$

Hence.

$$\int (x+4)\frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2(x+4) - 4(\sqrt{x}-1)e^{\sqrt{x}} + C$$

i.e. $$\boxed{\int (x+4)\frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = (2x - 4\sqrt{x}+12)e^{\sqrt{x}} + C}$$

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(2)(c) Short Method:

$$\int \frac{x^2\sqrt{x^2-3}}{x}\,dx = \int x\sqrt{x^2-3}\,dx = \frac{1}{2}\int 2x\cdot\sqrt{x^2-3}\,dx$$

$$= \frac{1}{2}\int \frac{d}{dx}\left(\frac{2}{3}(x^2-3)^{3/2}\right)dx.$$

$$= \frac{1}{2}\cdot\frac{2}{3}(x^2-3)^{3/2} + C,$$ by the Fundamental Theorem of Calculus

$$\boxed{\int \frac{x^2\sqrt{x^2-3}}{x}\,dx = \frac{1}{3}(x^2-3)^{3/2} + C}$$

Long Method.

$$\int \frac{x^2\sqrt{x^2-3}}{x}\,dx.$$

Take: $x = \sqrt{3}\sec\theta$

Hence: $x^2-3 = (\sqrt{3}\sec\theta)^2 - 3$

$= 3\sec^2\theta - 3$

$= 3(\sec^2\theta - 1)$

$= 3\tan^2\theta$

So that:

$$\frac{x^2\sqrt{x^2-3}}{x} = \frac{3\sec^2\theta\sqrt{3\tan^2\theta}}{\sqrt{3}\sec\theta} = \frac{3\sec^2\theta\sqrt{3}\tan\theta}{\sqrt{3}\sec\theta}$$

$$= 3\sec\theta\tan\theta.$$

On the other hand,

$$\frac{dx}{d\theta} = \sqrt{3}\frac{d}{d\theta}(\sec\theta) = \sqrt{3}\sec\theta\tan\theta.$$

Therefore..

$$\int\frac{x^2\sqrt{x^2-3}}{x}\,dx = \int(3\sec\theta\tan\theta)(\sqrt{3}\sec\theta\tan\theta)\,d\theta$$

$$= 3\sqrt{3}\int\sec^2\theta\tan^2\theta\,d\theta = 3\sqrt{3}\int\tan^2\theta\sec^2\theta\,d\theta$$

Let $y = \tan\theta$, $\frac{dy}{d\theta} = \sec^2\theta$. Then a!

$$= 3\sqrt{3}\int y^2\,dy = 3\sqrt{3}\,\frac{y^3}{3} + C = \sqrt{3}\,y^3 + C$$

$$= \sqrt{3}\tan^3\theta + C_1.$$

We found: $3\tan^2\theta = x^2-3.$

Then $\tan\theta = \sqrt{\frac{x^2-3}{3}} \Rightarrow \tan^3\theta = \left(\frac{x^2-3}{3}\right)^{3/2}$

Thus: $$\int\frac{x^2\sqrt{x^2-3}}{x}\,dx = \sqrt{3}\left(\frac{\sqrt{x^2-3}}{\sqrt{3}}\right)^3 + C_1.$$

$$= \frac{\sqrt{3}}{(\sqrt{3})^3}\left(\sqrt{x^2-3}\right)^3 + C_1$$

$$= S =$$

$$\boxed{\int \frac{x^2\sqrt{x^2-3}}{x}\,dx = \frac{1}{3}(x^2-3)^{3/2} + C}$$

Same result.

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② (d) $\displaystyle\int \frac{1}{(x^2+2x+5)(x^2-2x+4)}\,dx$.

Note that:

$$x^2+2x+5 = (x^2+2x+1)+4 = (x+1)^2+4$$

$$x^2-2x+4 = (x^2-2x+1)+3 = (x-1)^2+3$$

Hence:

$$\frac{1}{(x^2+2x+5)(x^2-2x+4)} = \frac{1}{((x+1)^2+4)}\cdot\frac{1}{((x-1)^2+3)} =$$

$$= \frac{A(x+1)+B}{(x+1)^2+4} + \frac{C(x+1)+D}{(x-1)^2+3}$$

$$= \frac{(A(x+1)+B)((x-1)^2+3) + (C(x-1)+D)((x+1)^2+4)}{((x+1)^2+4)((x-1)^2+3)}$$

Then, the equality:

$$(A(x+1)+B)((x-1)^2+3) + (C(x-1)+D)((x+1)^2+4) = 1$$

Should hold for all values of x.

In particular:

Take $\underline{x=1}$,

$$(2A+B)(3) + D(8) = 1$$

$$\Rightarrow \boxed{6A+3B+8D=1}$$

Take $\underline{x=-1}$,

$$B(7) + (-2C+D)4 = 1$$

$$\boxed{7B-8C+4D=1}$$

Take $\underline{x=0}$,

$$(A+B)(4) + (-C+D)5 = 1$$

$$\boxed{4A+4B-5C+5D=1}$$

Take $\underline{x=2}$.

$$(3A+B)(4) + (C+D)12 = 1$$

$$\boxed{12A+4B+12C+12D=1}$$

So, we have the system of equations for A, B, C and D:

$$\boxed{\begin{aligned} 6A+3B +8D &= 1 \\ 7B-8C+4D &= 1 \\ 4A+4B-5C+5D &= 1 \\ 12A+4B+12C+12D &= 1 \end{aligned}}$$

The solution to the system is:

$$A = \frac{13}{232} \; ; \; B = \frac{5}{116} \; ; \; C = -\frac{25}{464} \; ; \; D = \frac{31}{464}.$$

Hence, the integral becomes:

$$\int \frac{(13/232)(x+1) + (5/116)}{(x+1)^2+4} + \frac{(-25/464)(x-1) + (31/464)}{(x-1)^2+3}\, dx$$

$$= \frac{1}{232}\int \frac{13(x+1)+10}{(x+1)^2+4}\, dx + \left(\frac{-1}{464}\right)\int \frac{25(x-1)-31}{(x-1)^2+3}\, dx$$

We have to compute these two integrals:

$$\int \frac{13(x+1)+10}{(x+1)^2+4}\, dx = \frac{13}{2}\int \frac{2(x+1)}{(x+1)^2+4}\, dx + 10\int \frac{dx}{(x+1)^2+4}$$

$$= \frac{13}{2}\operatorname{Log}\left|(x+1)^2+4\right| + \frac{10}{2}\operatorname{Arctan}\left(\frac{x+1}{2}\right) + C.$$

$$\int \frac{25(x-1)-31}{(x-1)^2+3}\, dx = \frac{25}{2}\int \frac{2(x-1)}{(x-1)^2+3}\, dx - 31\int \frac{1}{(x-1)^2+3}\, dx$$

$$= \frac{25}{2}\operatorname{Log}\left|(x-1)^2+3\right| - \frac{31}{\sqrt{3}}\operatorname{Arctan}\left(\frac{x-1}{\sqrt{3}}\right) + C'.$$

Therefore:

$$\boxed{\int \frac{1}{(x^2+2x+5)(x^2-2x+4)}\, dx = \frac{13}{464}\operatorname{Log}\left((x+1)^2+4\right) + \frac{5}{232}\operatorname{Arctan}\left(\frac{x+1}{2}\right) + \left(\frac{-25}{928}\right)\operatorname{Log}\left((x-1)^2+3\right) + \frac{31}{\sqrt{3}}\operatorname{Arctan}\left(\frac{x-1}{\sqrt{3}}\right) + C'}$$

Q (2e). We need to compute $\int_0^T e^{-x}\cos x\,dx$,

then take limit when $T \longrightarrow \infty$.

We have the indefinite integral:

$$\int e^{-x}\cos x\,dx = -e^{-x}\cos x - \int (-e^{-x})(-\sin x)\,dx$$

$$= -e^{-x}\cos x - \int e^{-x}(\sin x)\,dx.$$

$$= -e^{-x}\cos x - \left[-e^{-x}\sin x - \int (-e^{-x})\cos x\,dx\right]$$

$$= -e^{-x}\cos x - \left[-e^{-x}\sin x + \int e^{-x}\cos x\,dx\right]$$

$$= -e^{-x}\cos x + e^{-x}\sin x - \int e^{-x}\cos x\,dx$$

i.e.

$$2\int e^{-x}\cos x\,dx = e^{-x}(\sin x - \cos x) + C'$$

i.e. $$\int e^{-x}\cos x\,dx = e^{-x}\frac{(\sin x - \cos x)}{2} + C$$

Therefore the definite integral is:

$$\int_0^T e^{-x}\cos x\,dx = e^{-T}\frac{\sin(T) - \cos(T)}{2} - \left(\frac{0-1}{2}\right).$$

So that:

$$\int_0^\infty e^{-x}\cos x\,dx = \lim_{T\to\infty}\int_0^T e^{-x}\cos x\,dx =$$

$$= \lim_{T \to \infty} \left[e^{-T} \frac{(\sin T - \cos T)}{2} + \frac{1}{2} \right]$$

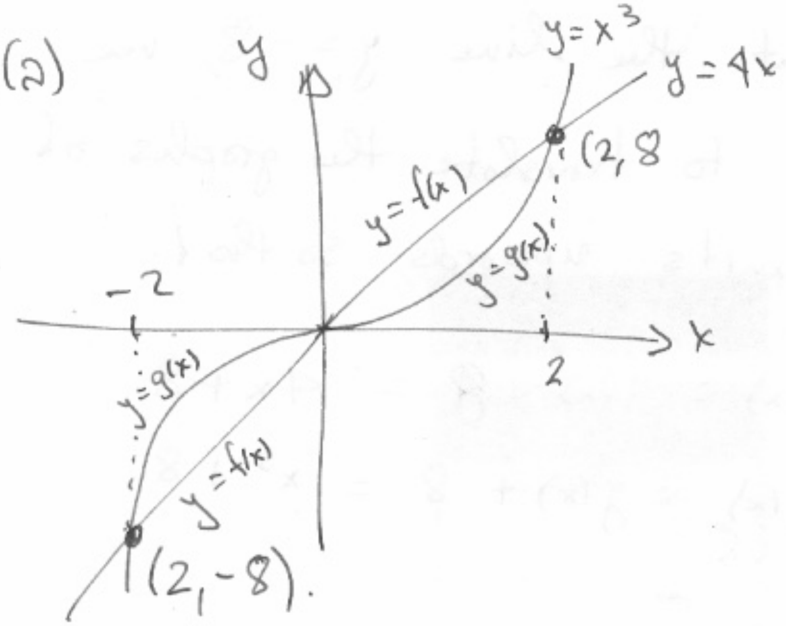
Since $\lim_{T \to \infty} e^{-T} \sin T = 0$

$\lim_{T \to \infty} e^{-T} \cos T = 0$;

$$= 0 + \frac{1}{2}$$

Therefore $\boxed{\int_0^\infty e^{-x} \cos x \, dx = \frac{1}{2}}$

3 (a)



The intersection points satisfy the equation:

$$g(x)=f(x)$$

$$x^3=4x$$

$$x^3-4x=0$$

$$x(x^2-4)=0$$

$$x(x-2)(x+2)=0$$

Then, $x_1=0$, $x_2=+2$, $x_3=-2$.

The area is:

$$A=\int_{-2}^{2}|f(x)-g(x)|\,dx \;=2\int_0^2|f(x)-g(x)|\,dx,$$ by the symmetry.

$$=2\int_0^2 f(x)-g(x)\,dx,$$ since $f(x)-g(x)\geq 0$

i.e. $f(x)\geq g(x)$

$$=2\int_0^2(4x-x^3)\,dx.$$

i.e. $f(x)$ is above $g(x)$.

$$=2\left(2x^2-\frac{1}{4}x^4\right)\Big|_0^2=2\left(2(2)^2-\frac{1}{4}(2)^4\right)$$

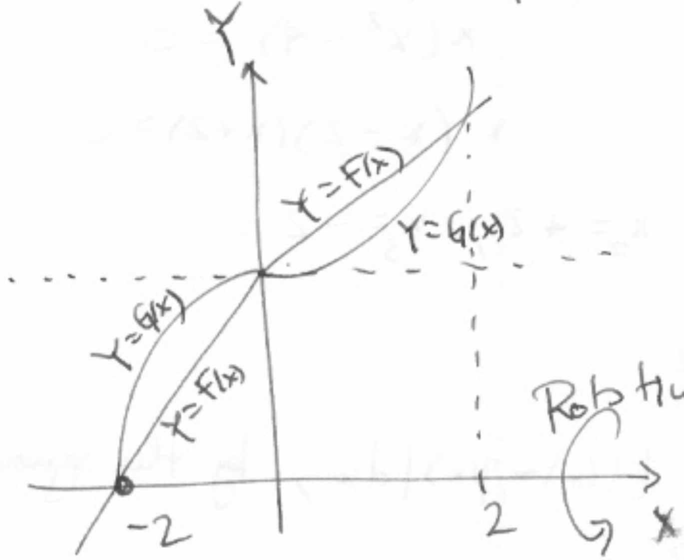
$$=2\left(2^3-\frac{2^4}{2^2}\right)=2\left(2^3-2^2\right)=2^3(2-1)=2^3=8$$

$\boxed{\text{Area}=8}$

3(b). To rotate about the line $y = -8$, we see it is equivalent to translate the graphs of the function 8 units upwards, so that:

$$Y = F(x) = f(x) + 8 = 4x + 8$$

$$Y = G(x) = g(x) + 8 = x^3 + 8$$



Then, the volume is:

$$V = \pi \int_{-2}^{2} \left| F^2(x) - G^2(x) \right| dx$$

Hence:

$$V = \pi \int_{-2}^{0} \left(G^2(x) - F^2(x) \right) dx + \pi \int_{0}^{2} \left(F^2(x) - G^2(x) \right) dx$$

$$= \pi \int_{-2}^{0} \left((x^3+8)^2 - (4x+8)^2 \right) dx + \pi \int_{0}^{2} (4x+8)^2 - (x^3+8)^2 \, dx$$

There is <u>no</u> symmetry here, since the region in between $[0,2]$ rotates <u>more</u> distance than the region that is in the interval $[-2,0]$.

$$= \pi \int_{-2}^{0} (x^6 + 16x^3 - 16x^2 - 64x)\, dx \; +$$

$$+ \pi \int_{0}^{2} (-x^6 - 16x^3 + 16x^2 + 64x)\, dx$$

$$= -\pi \left(\frac{x^7}{7} + 4x^4 - \frac{16}{3}x^3 - 32x^2 \right) \Bigg|_{0}^{-2}$$

$$+ \pi \left(-\frac{x^7}{7} - 4x^4 + \frac{16}{3}x^3 + 32x^2 \right) \Bigg|_{0}^{2}.$$

$$= -\pi \left(\cancel{-\frac{2^7}{7}} + 4\cdot 2^4 + \cancel{\frac{16}{3}\cdot 2^3} - 32\cdot 4 \right) + \pi \left(\cancel{-\frac{2^7}{7}} - 4\cdot 2^4 + \cancel{\frac{16}{3} 2^3} + 32\cdot 4 \right)$$

$$= -\pi \left(2^6 - 2^7 \right) + \pi \left(-2^6 + 2^7 \right) = \pi \left(2^7 - 2^6 \right) + \pi \left(2^7 - 2^6 \right)$$

$$= 2\pi \left(2^7 - 2^6 \right) = 2 \cdot 2^6 \pi (2-1) = 2^7 \pi.$$

$$\boxed{\text{Volume} = 128\pi}$$