UAM - Azcapotzalco. Intro al Cálculo. | Miércoles 14 de marzo 2018.

Quiz #8 Nombre: Answer Key.

Justifique y Argumente sus respuestas, Muestre todas sus cuentas. Explique mientras hace su desarrollo. Ordene su trabajo e ideas

① ¿Para qué valor de b, la función $g(x)$ es continua en $x=0$?

$$g(x) = \begin{cases} \dfrac{x-b}{b+1}, & x<0 \\ x^2+b, & x \geq 0 \end{cases}$$

② Explique por qué la ecuación $\cos x = x + \frac{1}{2}$ tiene (al menos) una solución

③ Encuentre las asíntotas de la función:

$$h(x) = \frac{7x^3}{x^3 - 5x^2 + 6x}$$

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ANSWER KEY.

(1) The only chance for $g(x)$ to be discontinuous is at $x=0$, where the intervals of definition of g change.

Then, let us use the definition of continuity at $x=0$.

(i) $g(0) = 0^2 + b = b$.

(ii) Compute the side limits:

$$\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} \frac{x-b}{b+1} = \frac{-b}{b+1}$$

$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} x^2+b = b$$

For the limit to exist, these limits should coincide

$$b = \frac{-b}{b+1}$$

Then: $b(b+1) = -b \Rightarrow (b^2+b)+b = 0 \Rightarrow b^2+2b=0$

$\Rightarrow b(b+2)=0$ Then, there are two values of b:

$b_1 = 0$ and $b_2 = -2$.

Then: $$\lim_{x\to 0} g(x) = \frac{-b}{b+1} = b = \begin{cases} 0 \\ -2 \end{cases}$$

They are equal

(iii) From the previous equation:

$$\lim_{x\to 0} g(x) = \frac{-b}{b+1} = b \overset{\text{by (i)}}{=} g(0)$$

Then, $g(x)$ is continuous for $b=0$ or $b=-2$.

We then have two possible functions

$$\boxed{g_1(x) = \begin{cases} x, & x<0 \\ x^2, & x\geq 0 \end{cases}} \quad \text{or} \quad \boxed{g_2(x) = \begin{cases} -(x+2), & x<0 \\ x^2-2, & x\geq 0 \end{cases}}$$

(2) We have the equation: $\cos x = x + \frac{1}{2}$.

Which is equivalent to: $\cos x - x = \frac{1}{2}$

Let $f(x) = \cos x - x$. Then, the equation is:

$$f(x) = \frac{1}{2}.$$

Now, the function $f(x) = \cos x - x$ is continuous in $\mathbb{R}$, because $\cos x$ and $-x$ are continuous in $\mathbb{R}$.

Now:
$$f(0) = \cos(0) - 0 = 1$$
$$f\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} - \frac{\pi}{2} = 0 - \frac{\pi}{2}.$$

Then:
$$1 = f(0) > \frac{1}{2} > f\left(\frac{\pi}{2}\right) = 0$$

There, by the __intermediate value theorem__, there must be a $c \in \left(0, \frac{\pi}{2}\right)$ such that:

$$f(c) = \frac{1}{2}.$$ then: $f(0) > f(c) > f\left(\frac{\pi}{2}\right)$

i.e. there exists $c \in \left(0, \frac{\pi}{2}\right)$, such that:

$$\cos(c) = c + \frac{1}{2}$$

i.e., the equation has (at least) one root.

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(3) We must find where $h(x)$ is not defined, and take limits there. Also, take limits to $\pm\infty$.

Notice that: $$h(x) = \frac{7x^3}{x(x^2-5x+6)}$$

i.e.

$$h(x) = \frac{7x^3}{x(x-2)(x-3)}$$

Then, $\mathrm{Dom}(h) = \mathbb{R}\setminus\{0,2,3\}$.

We must take:

(a) $\lim_{x\to 0^{\pm}} h(x)$, (b) $\lim_{x\to 2^{\pm}} h(x)$ and (c) $\lim_{x\to 3^{\pm}} h(x)$.

(a) Notice that, for $x \neq 0$:

$$h(x) = \frac{7x^3}{x(x-2)(x-3)} = \frac{7x^2}{(x-2)(x-3)}$$

Hence:

$$\lim_{x\to 0} h(x) = \lim_{x\to 0} \frac{7x^2}{(x-2)(x-3)} = \frac{0}{(-2)(-3)} = 0.$$

Then, $x=0$ is <u>not</u> an asymptote.

(b) $$\lim_{x\to 2^{\pm}} h(x) = \lim_{x\to 2^{\pm}} \frac{7x^2}{(x-2)(x-3)} = \lim_{x\to 2^{\pm}} \frac{7x^2}{(x-3)} \lim_{x\to 2^{\pm}} \frac{1}{x-2}$$

$$= \left(\frac{7\cdot 4}{2-3}\right) \lim_{x\to 2^{\pm}} \frac{1}{x-2} = (-28) \lim_{x\to 2^{\pm}} \frac{1}{x-2}$$

$= \mp\infty$. Then $\boxed{x=2}$ is a <u>vertical asymptote</u>.

(c). $\lim_{x \to 3^{\pm}} h(x) = \lim_{x\to 3^{\pm}} \frac{7x^2}{(x-2)(x-3)} = \lim_{x\to 3^{\pm}} \frac{7x^2}{(x-2)} \cdot \lim_{x\to 3^{\pm}} \frac{1}{x-3}$

$= \left(\frac{7\cdot 9}{3-2}\right) \lim_{x\to 3^{\pm}} \frac{1}{x-3} = 63 \lim_{x\to 3^{\pm}} \frac{1}{x-3}$.

$= \pm\infty$. Then $y = \boxed{x=3}$ is a second vertical asymptote.

Now,

$$\lim_{x\to\pm\infty} h(x) = \lim_{x\to\pm\infty} \frac{7x^3}{x^3-5x^2+6x} = \lim_{x\to\pm\infty} \frac{x^3}{x^3} \frac{7}{\left(1-\frac{5}{x}+\frac{6}{x^2}\right)}$$

$$= \lim_{x\to\pm\infty} \frac{7}{\left(1-\frac{5}{x}+\frac{6}{x^2}\right)} = \frac{7}{1-0+0} = 7$$

Then $\boxed{y=7}$ is the only one horizontal asymptote.