UAM-Azcapotzalco ECUACIONES DIFERENCIALES | M. 14 de marzo de 2018.

Quiz #8 : Nombre: KEY.

JUSTIFIQUE y ARGUMENTE sus respuestas. Muestre todas sus cuentas. EXPLIQUE mientras hace su desarrollo. Ordene su trabajo e ideas

① . Encuentre la solución particular de:

$$3\frac{dy}{dt} + 6y = 12t\,e^{-3t}.$$

② Encuentre la forma de la solución particular de.

$$\ddot{y} - 2\dot{y} + y = 4\sin t + 18e^{2t} + t^2e^{-t}$$

③ Encuentre la solución particular de.

$$\ddot{y} + y = \tan t$$

ECUACIONES DIFERENCIALES Quiz 8. Miércoles, 14 de marzo de 2018.

ANSWER KEY

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(1) We must solve:

$$3\frac{dy}{dt} + 6y = 12te^{-3t}.$$

(a) We must find the solution to the homogeneous eq'n:

$$3\dot{y}_h + 6y_h = 0 \quad ;$$

i.e. $\dot{y}_h + 2y_h = 0 \Rightarrow y_h(t) = e^{-2t}$.

(b) Now, a first trial function for the particular sol'n is.

$$\tilde{y}_p(t) = (\alpha t + \beta)e^{-3t}.$$

Notice that this proposal do <u>not</u> repeat solutions to the homogeneous equation. Then,

$$y_p(t) = (\alpha t + \beta)e^{-3t}$$

is the right trial. Then, substitute into the Diff. Eq'n:

$$3\dot{y}_p + 6y_p = 12te^{-3t}.$$

$$\dot{y}_p + 2y_p = 4te^{-3t}.$$

$$\left(\alpha e^{-3t} - 3(\alpha t + \beta)e^{-3t}\right) + 2\left((\alpha t + \beta)e^{-3t}\right) = 4te^{-3t}.$$

i.e. $\left((\alpha - 3\beta) - 3\alpha t\right)e^{-3t} + 2(\alpha t + \beta)e^{-3t} = 4te^{-3t}$

i.e. $\left((\alpha - 3\beta) - 3\alpha t\right) + (2\alpha t + 2\beta) = 4t$

i.e. $\underbrace{(\alpha - \beta)}_{=1 =} + (-\alpha)t = 4t.$

Then $\alpha - \beta = 0$ $\Rightarrow$ $\alpha = \beta$

$-\alpha = 4$ $\boxed{\alpha = -4}$ $\Rightarrow$ $\boxed{\beta = -4}$

Then, the particular solution is:

$$\boxed{y_p(t) = -4(t+1)e^{-3t}}$$

The general solution is:

$$\boxed{y(t) = Ce^{-2t} - 4(t+1)e^{-3t}}$$

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(2). The equation is:

$$\ddot{y} - 2\dot{y} + y = 4\sin t + 18e^{2t} + t^2 e^{-t}.$$

(a) First, we must solve the homogeneous eq'n.

$$\ddot{y}_h - 2\dot{y}_h + y_h = 0.$$

Since it is linear, homogeneous and constants coeff's: $y_h = e^{rt}$.

Then: $r^2 - 2r + 1 = 0$

$(r-1)^2 = 0.$ $\Rightarrow$ $r_1 = r_2 = 1$

The solution to the homogeneous equation is:

$$y_h(t) = C_1 e^t + C_2 t e^t = (C_1 + C_2 t)e^t$$

(b) To each non-homogeneous term, we have:

(i) $y_{p,1}(t) = A\cos t + B\sin t.$

(ii) $y_{p,2}(t) = \tilde{C} e^{2t}$

(iii) $y_{p,3}(t) = (Dt^2 + Et + F)e^{-t}.$

Since none of the trial repeats a solution to the homogeneous equation, then, by the superposition principle, the general solution is:

$$\boxed{y(t) = (C_1 + C_2 t)e^{t} + A\cos t + B\sin t + \tilde{C}e^{2t} + (Dt^2 + Et + F)e^{-t}}$$

Being $y_p(t) = A\cos t + B\sin t + \tilde{C}e^{2t} + e^{-t}(Dt^2 + Et + F)$ the particular solution, being $A, B, \tilde{C}, D, E$ constants to be determined.

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(3). Here, we must find the particular solution by the method of Variation of Parameters.

$$\ddot{y} + y = \tan t.$$

(a) We must first solve the homogeneous equation:

$$\ddot{y}_h + y_h = 0.$$

with: $\boxed{\begin{aligned} y_1(t) &= \cos t \\ y_2(t) &= \sin t \end{aligned}}$

$$\Rightarrow \quad y_h(t) = C_1 y_1(t) + C_2 y_2(t)$$
$$= C_1 \cos t + C_2 \sin t$$

(b) Now, the particular solution has the form:

$$y_p(t) = A(t)\cos(t) + B(t)\sin t.$$

with

$$A(t) = -\int \frac{y_2(t)\, g(t)}{a(t)\, W[y_1, y_2](t)}\, dt \;;\; B(t) = \int \frac{y_1(t)\, g(t)}{a(t)\, W[y_1, y_2](t)}\, dt.$$

Here: $a(t) = 1$

$g(t) = \tan t$

and the wronskian:

$$W[y_1, y_2](t) = \det \begin{pmatrix} y_1(t) & y_2(t) \\ \dot{y}_1(t) & \dot{y}_2(t) \end{pmatrix} = \det \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

$$= \cos^2 t - (-\sin^2 t) = \cos^2 t + \sin^2 t = 1.$$

Hence:

$$A(t) = -\int \frac{(\sin t)(\tan t)}{1 \cdot 1} dt = -\int \frac{\sin^2 t}{\cos t} dt = -\int \frac{\sin^2 t}{\cos^2 t} \cos t \, dt$$

$$= -\int \frac{\sin^2 t}{1-\sin^2 t} \cos t \, dt \underset{y = \sin t}{=} -\int \frac{y^2}{1-y^2} dy = \int \frac{-y^2}{1-y^2} dy$$

$$= \int \frac{1-y^2-1}{1-y^2} dy = \int \left(1 - \frac{1}{1-y^2}\right) dy = y - \int \frac{1}{1-y^2} dy$$

$$= y - \int \frac{1/2}{1-y} - \frac{1/2}{1+y} dy = y - \frac{1}{2}\left(-\log|1-y| - \log|1+y|\right)$$

$$= y + \log\sqrt{1-y^2} = \sin t + \log|\cos t|$$

Also

$$B(t) = \int \frac{\cos t \tan t}{1 \cdot 1} dt = \int \sin t \, dt = -\cos t.$$

Hence:

$$y_p(t) = \left(\sin t + \log|\cos t|\right)\cos t + (-\cos t)\sin t.$$

i.e. $\boxed{y_p(t) = (\cos t)(\log|\cos t|)}$