UNIVERSIDAD AUTÓNOMA METROPOLITANA - AZCAPOTZALCO
INTRODUCCIÓN AL CÁLCULO
TRIMESTRE: INVIERNO DE 2018.

EXAMEN # 2.
FECHA: LUNES 19 DE MARZO DE 2018.

Nombre: ANSWER KEY.

Instrucciones:

- El examen consta de **CINCO** problemas, cada uno de 20 puntos, más dos problemas extra.
- Tienen **una** hora con **veinte (20)** minutos para resolverlos.
- Por favor **apaguen sus celulares**. Eviten la pena de quitarles sus exámenes.
- Para recibir puntaje, escriba de forma clara y concisa. Entregue su trabajo limpio y con sus ideas en orden. SIMPLIFIQUE. Muestre sus cuentas. **EXPLIQUE, ARGUMENTE y JUSTIFIQUE** sus respuestas.
- Problema **SIN explicación, desarrollo, justificación o argumento** vale **CERO** puntos.

PROBLEMAS

(1) **(20 puntos.)** Si $x \in (-\pi, -\pi/2)$ y si $\tan x = 3/2$, encuentre $\sin x$ y $\cos x$.

(2) **(20 puntos.)** Calcule el límite $\lim_{x\to 1} \sqrt{\sec^2\left(\frac{\pi}{3}x\right) + \frac{5}{\sqrt{3}}\tan\left(\frac{\pi}{3}x\right)}$.

(3) **(20 puntos.)** Calcule $\lim_{x\to -7}(x-3)\dfrac{|x+7|}{x+7}$.

(4) **(20 puntos.)** Para qué valor (o valores) de a, la función $F(x)$ es contínua en $x = 0$?

$$F(x) = \begin{cases} x^2 - a, & x \le 0, \\ \dfrac{-x+a}{1-a} & x > 0. \end{cases}$$

(5) **(20 puntos.)** Encuentre las asíntotas de la función

$$G(x) = \frac{x^2 + x + 1}{2x^2 - 32}.$$

(6) **(20 puntos extra.)** Calcule $\lim_{x\to\infty}(\sqrt{x+9} - \sqrt{x+4})$.

(7) Una función está dada por $H(x) = \begin{cases} 0, & x < -1, \\ x+1 & -1 \le x \le 0 \\ -x+1 & \cancel{1 \le x \le 0} \\ 0 & x > 1. \end{cases}$ $\leftarrow$ $0 \le x \le 1$

(a) **(10 puntos extra.)** Esboce la gráfica de $H(x)$.
(b) **(10 puntos extra.)** Esboce la curva $-2H(2x) + 2$.

(*) **FÓRMULAS**. Algunas de las siguientes fórmulas pueden serle de utilidad.

(a) $\sin\frac{\pi}{3} = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$; $\cos\frac{\pi}{3} = \sin\frac{\pi}{6} = \frac{1}{2}$; $\sin\frac{\pi}{2} = \cos\frac{\pi}{2} = \frac{\sqrt{2}}{2}$
(b) $\sin 2\theta = 2\sin\theta\cos\theta$
(c) $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$
(d) $\sin^2\alpha + \cos^2\alpha = 1$.
(e) $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\cos\beta$
(f) $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
(g) $\cos^2\alpha = \frac{1+\cos 2\alpha}{2}$.
(h) $\sin^2\alpha = \frac{1-\cos 2\alpha}{2}$.

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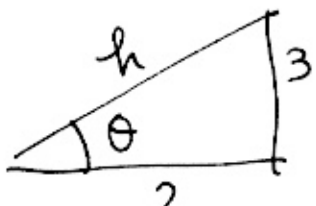
(1) We first compute the magnitude of $\sin x$ and $\cos x$ by using a right triangle.



Here, the angle θ is such that:

$$\tan\theta = \frac{3}{2}.$$

By the Pythagorean theorem:

$$h = \sqrt{3^2+2^2} = \sqrt{13}$$

Then: $\sin\theta = \frac{3}{\sqrt{13}}$ and $\cos\theta = \frac{2}{\sqrt{13}}$, $\theta \in \left(0, \frac{\pi}{2}\right)$

Now, for $x \in \left(-\pi, -\frac{\pi}{2}\right)$, sin and $\cos x < 0$, but with the same magnitude as for θ. Then:

$$\boxed{\sin x = \frac{-3}{\sqrt{13}}} \quad \text{and} \quad \boxed{\cos x = \frac{-2}{\sqrt{13}}}$$

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(2) The functions $\cos(\theta)$, then $\sec(\theta)$ and $\tan(\theta)$ are continuous at $\theta = \frac{\pi}{3}$, so the composite function

$$\sqrt{\sec^2\theta + \frac{5}{\sqrt{3}}\tan\theta}$$

is continuous at $\theta = \frac{\pi}{3}$.

Hence:

$$\lim_{x \to 1} \sqrt{\sec^2\left(\frac{\pi}{3}x\right) + \frac{5}{\sqrt{3}} \tan\left(\frac{\pi}{3}x\right)} =$$

$$= \sqrt{\left(\lim_{x \to 1} \sec\left(\frac{\pi}{3}x\right)\right)^2 + \frac{5}{\sqrt{3}}\left(\lim_{x \to 1} \tan\left(\frac{\pi}{3}x\right)\right)}$$

$$= \sqrt{\left(\sec \frac{\pi}{3}\right)^2 + \frac{5}{\sqrt{3}} \tan\left(\frac{\pi}{3}\right)}$$

Now $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$, $\cos \frac{\pi}{3} = \frac{1}{2}$

Hence $\tan\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$ and $\sec\left(\frac{\pi}{3}\right) = \frac{1}{\cos\left(\frac{\pi}{3}\right)} = 2$

Therefore:

$$\lim_{x \to 1} \sqrt{\left(\sec^2\left(\frac{\pi}{3}x\right) + \frac{5}{\sqrt{3}} \tan\left(\frac{\pi}{3}x\right)\right.} =$$

$$= \sqrt{\left(\sec \frac{\pi}{3}\right)^2 + \frac{5}{\sqrt{3}} \tan\left(\frac{\pi}{3}\right)} = \sqrt{2^2 + \frac{5}{\sqrt{3}}\sqrt{3}}$$

$$= \sqrt{4 + 5} = \sqrt{9}.$$

and so:

$$\boxed{\lim_{x \to 1} \sqrt{\sec^2\left(\frac{\pi}{3}x\right) + \frac{5}{\sqrt{3}} \tan\left(\frac{\pi}{3}x\right)} = 3}$$

(3) We should compute the side limits.

$$\lim_{x\to -7^-} (x-3)\frac{|x+7|}{x+7} = \lim_{x\to -7^-} (x-3)\frac{(-(x+7))}{x+7} =$$

(since $x<-7$, then $x+7<0$).

$$= \lim_{x\to -7^-} (x-3)(-1) = (-7-3)(-1) = 10$$

$$\lim_{x\to -7^+} (x-3)\frac{|x+7|}{x+7} = \lim_{x\to -7^+} (x-3)\frac{(x+7)}{x+7}$$

(since $-7<x$, then $x+7>0$)

$$= \lim_{x\to -7^+} (x-3) = -7-3 = -10.$$

The side limits are different, then <u>the limit does **not** exist.</u>

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(4) We should compute: (a) $F(0)$

(b) $\lim_{x\to 0} F(x)$

(c) Check if results in (a) and (b) coincide.

If (a), (b) and (c) holds, $F(x)$ is continuous at $x=0$.

(a) $F(0) = (x^2-a)\big|_{x=0} = -a$

(b) We should compute the side limits:

$$\lim_{x\to 0^-} F(x) = \lim_{x\to 0^-} x^2 - a = -a.$$

This coincides with $F(0)$.
$=3$

$$\lim_{x\to 0^+} F(x) = \lim_{x\to 0^+} \frac{-x+a}{1-a} = \frac{a}{1-a} \quad (\text{if } a \neq 1),$$

For the limits to exist, the side limits should be the same:

$$-a = \frac{a}{1-a}$$

i.e.

$$-a(1-a) = a$$

$$a^2 - a = a$$

$$a^2 - 2a = 0 \Rightarrow a(a-2) = 0 \Rightarrow \begin{cases} a_1 = 0 \\ \text{or} \\ a_2 = 2 \end{cases}$$

Then, the limit exists if:

$$a = a_1 = 0$$

or

$$a = a_2 = 2.$$

$$\lim_{x\to 0} F(x) = -a = \frac{a}{1-a} = \begin{cases} 0, & \text{if } a = 0 \\ -2, & \text{if } a = 2 \end{cases}$$

(c) Since $F(0) = \lim_{x\to 0^-} F(x)$ and $\lim_{x\to 0^-} F(x) = \lim_{x\to 0} F(x)$

then

$$F(0) = \lim_{x\to 0} F(x).$$

i.e.

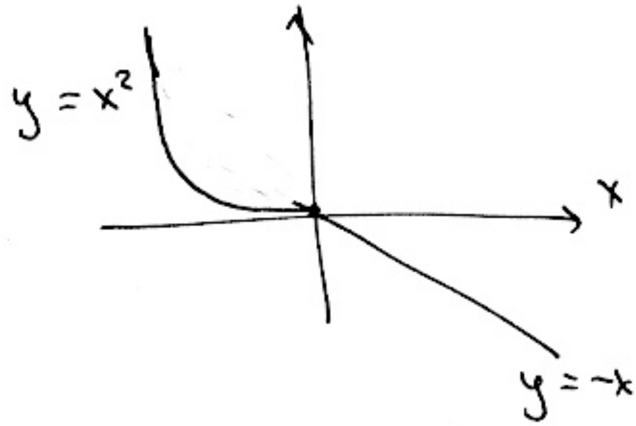
$$\begin{cases} F(0) = \lim_{x\to 0} F(x) = 0, & \text{if } a = 0 \\ F(0) = \lim_{x\to 0} F(x) = -2, & \text{if } a = 2 \end{cases}$$

Then, $F(x)$ is continuous at $x=0$ if $a=0$ or $a=2$.

Then, we have two different functions which are continuous at $x=0$.

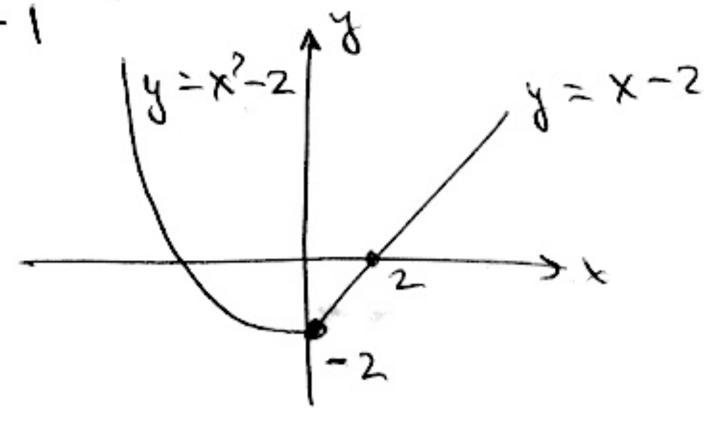
For $a=0$:

$$F(x)=\begin{cases} x^2, & x\le 0 \\ -x, & x>0 \end{cases}$$



For $a=2$:

$$F(x)=\begin{cases} x^2-2, & x\le 0 \\ \dfrac{-x+2}{1-2}=\dfrac{-(x-2)}{-1}=x-2, & \text{if } x>0 \end{cases}$$



(5) Find the asymptotes of the function:

$$f(x)=\frac{x^2+x+1}{2x^2-32}$$

We can write:

$$f(x)=\frac{x^2+x+1}{2(x^2-16)}=\frac{x^2+x+1}{2(x-4)(x+4)}$$

Candidates for vertical asymptotes:

$x=4$

$x=-4$.

$$\lim_{x\to 4^+} F(x) = \lim_{x\to 4^+} \frac{x^2+x+1}{2(x-4)(x+4)} = \lim_{x\to 4^+} \frac{x^2+x+1}{2(x+4)} \lim_{x\to 4^+} \frac{1}{x-4}$$

$$= \frac{21}{16} \lim_{x\to 4^+} \frac{1}{x-4} = +\infty$$

$$\lim_{x\to 4^-} F(x) = \frac{21}{16} \lim_{x\to 4^-} \frac{1}{x-4} = -\infty$$

Then $\boxed{x=4 \text{ is a vertical asymptote}}$

$$\lim_{x\to -4^+} F(x) = \lim_{x\to -4^+} \frac{x^2+x+1}{2(x-4)} \lim_{x\to -4^+} \frac{1}{x+4}$$

$$= \frac{13}{-16} \lim_{x\to -4^+} \frac{1}{x+4} = \left(-\frac{13}{16}\right)(+\infty) = -\infty$$

$$\lim_{x\to -4^-} F(x) = \left(-\frac{13}{16}\right) \lim_{x\to -4^-} \frac{1}{x+4} = \left(-\frac{13}{16}\right)(-\infty) = +\infty.$$

Then $\boxed{x=-4 \text{ is a vertical asymptote}}$

Finally:

$$\lim_{x\to \pm\infty} F(x) = \lim_{x\to \pm\infty} \frac{x^2\left(1+\frac{1}{x}+\frac{1}{x^2}\right)}{x^2\left(2-\frac{32}{x^2}\right)} = \lim_{x\to \pm\infty} \frac{1+\frac{1}{x}+\frac{1}{x^2}}{2-\frac{32}{x^2}}$$

$$= \frac{1+0+0}{2-0} = \frac{1}{2}.$$

Then $\boxed{y=\frac{1}{2} \text{ is a horizontal asymptote}}$

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(6) Compute the limit:

$$\lim_{x\to\infty}\left(\sqrt{x+9}-\sqrt{x+4}\right) = \lim_{x\to\infty}\left(\frac{\sqrt{x+9}-\sqrt{x+4}}{\sqrt{x+9}+\sqrt{x+4}}\right)\cdot\left(\sqrt{x+9}+\sqrt{x+4}\right)$$

$$= \lim_{x\to\infty}\frac{(x+9)-(x+4)}{\sqrt{x+9}+\sqrt{x+4}} = \lim_{x\to\infty}\frac{5}{\sqrt{x+9}+\sqrt{x+4}}$$

$$= \lim_{x\to\infty}\frac{1}{\sqrt{x}}\left(\frac{5}{\sqrt{1+\frac{9}{x}}+\sqrt{1+\frac{4}{x}}}\right) = \left(\lim_{x\to\infty}\frac{1}{\sqrt{x}}\right)\left(\frac{5}{\sqrt{1}+\sqrt{1}}\right)$$

$$= 0\cdot\left(\frac{5}{2}\right) = 0.$$

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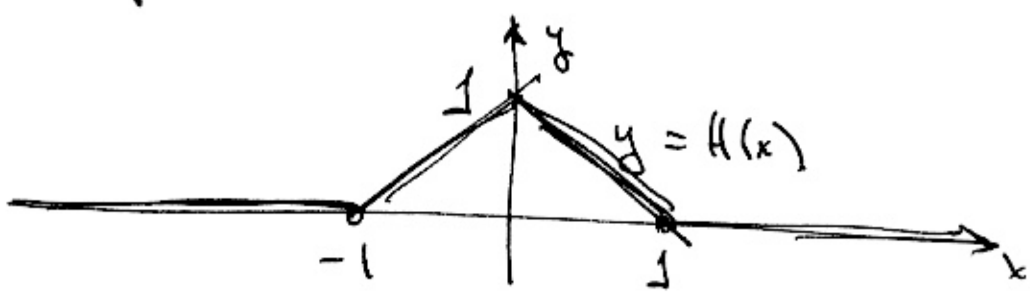
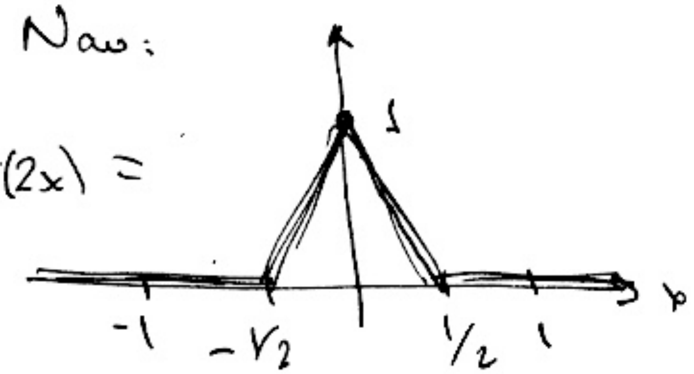
(7) The graph of $H(x)$ is:

(a)

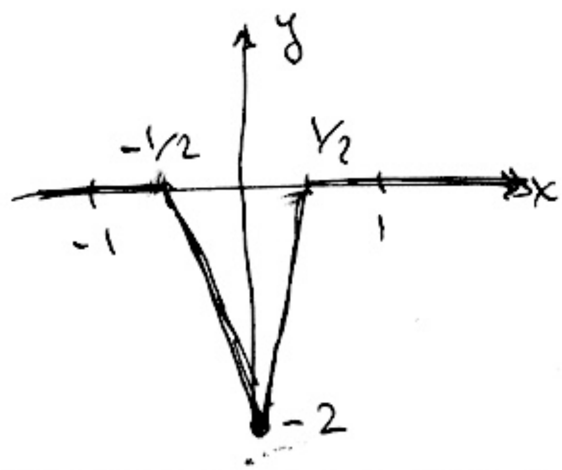


(b) Now:

$H(2x) =$



$-2H(2x) =$



Finally.

$-2H(2x)+2 =$