UNIVERSIDAD AUTÓNOMA METROPOLITANA - AZCAPOTZALCO
ECUACIONES DIFERENCIALES ORDINARIAS
TRIMESTRE: INVIERNO DE 2018.

EXAMEN # 2.
FECHA: **LUNES 19 DE MARZO DE 2018**

Nombre: ANSWER KEY

Instrucciones:

- El examen tiene **CUATRO** problemas de 25 puntos cada uno y otro de 25 puntos extra.
- Tienen **una** hora con **veinte (20)** minutos para resolverlos.
- Por favor **apaguen sus celulares**. Evíten la pena de quitarles sus exámenes.
- Para recibir puntaje, escriba de forma clara y concisa. Entregue su trabajo limpio y con sus ideas en orden. SIMPLIFIQUE. Muestre sus cuentas. **EXPLIQUE, ARGUMENTE y JUSTIFIQUE** sus respuestas.
- Problema **SIN explicación, desarrollo, justificación o argumento** vale **CERO** puntos.

PROBLEMAS

(1) **(25 puntos.)** Resuelva la ecuación diferencial.

$$(t - t^2 y)\frac{dy}{dt} = 3t^2 + y.$$

(2) **(25 puntos.)** Resuelva el problema de valores iniciales:

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = 0 \qquad y(0) = 0, \qquad \frac{dy}{dt}(0) = -1.$$

(3) **(25 puntos.)** Usando el método de la conjetura sensata, encuentre una solución particular de la ecuación diferencial:

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = 8e^{-2t}.$$

(4) **(25 puntos.)** Usando el método de variación de parámetros, encuentre una solución particular de la ecuación diferencial:

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = 8e^{-2t}.$$

(5) **(25 puntos extra.)** Si la ecuación diferencial tiene por solución:

$$y(t) = Ce^{-3t} + 3t^2e^{-3t},$$

encuentre la ecuación diferencial.

UAM- Azcapotzalco Ecuaciones Diferenciales Ordinarias

Examen #2 ANSWER KEY. Lunes, Marzo 19, 2018

(1) The Diff. Eq'n could be written as:

$$-(3t^2+y) + (t-t^2y)\frac{dy}{dt} = 0$$

or

$$(3t^2+y) + (t^2y-t)\frac{dy}{dt} = 0.$$

If $M(t,y) = 3t^2+y$

$N(t,y) = t^2y-t$, we can write:

$$M(t,y) + N(t,y)\frac{dy}{dt} = 0$$

To check if this Diff Eq'n is exact, we must compute:

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(3t^2+y) = 1$$

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial t}(t^2y-t) = 2ty-1$$

Since they are different, we must multiply by an integrating factor: $\mu(t,y)$:

$$\mu M + \mu N\frac{dy}{dt} = 0$$

To see if this Diff. Eq is exact, we must compute:

$\frac{\partial}{\partial y}(\mu M) = \mu_y M + \mu M_y$

$\frac{\partial}{\partial t}(\mu N) = \mu_t N + \mu N_t$

and they should coincide:

$$\mu_y M + \mu M_y = \mu_t N + \mu N_t$$

Therefore, assuming: $\mu_y = 0$:

$$\mu_t = \frac{(M_y - N_t)}{N}\mu.$$

But $M_y - N_t = (1) - (2ty - 1) = 1 - 2ty + 1$
$= 2 - 2ty = 2(1 - yt)$

Now $\dfrac{M_y - N_t}{N} = \dfrac{2(1-yt)}{t^2y - t} = \dfrac{2(1-yt)}{t(ty-1)} = \dfrac{-2}{t}$

Then:

$$\mu_t = -\frac{2}{t}\mu \Rightarrow \mu(t) = e^{-2\int \frac{1}{t}dt} = e^{-2\log t}$$

$$\Longrightarrow \mu(t) = t^{-2}$$

Hence:

$$t^{-2}(3t^2+y) + t^{-2}(t^2y - t)\frac{dy}{dt} = 0$$

is exact, i.e.:

$$\frac{\partial \Psi}{\partial t} = \left(3 + \frac{y}{t^2}\right) \quad \text{and} \quad \frac{\partial \Psi}{\partial y} = \left(y - \frac{1}{t}\right) \cdots\cdots (*)$$

Integrating with respect to t the Diff Eq'n $\frac{\partial \Psi}{\partial t} = 3 + \frac{y}{t^2}$,

we get
$$\Psi(t,y) = 3t - \frac{y}{t} + f(y)$$

and differentiating with respect to y:

$$\frac{\partial \Psi}{\partial y} = -\frac{1}{t} + f'(y)$$

and comparing with the expression $\frac{\partial \Psi}{\partial y}$, in eq'n $(*)$ above:

$$y - \frac{1}{t} = -\frac{1}{t} + f'(y) \Rightarrow f'(y) = y \Rightarrow f(y) = \frac{1}{2}y^2 + \tilde{C}$$

Hence: $\Psi(t,y) = 3y - \frac{y}{t} + \frac{1}{2}y^2 + \tilde{C}$.

Since $\Psi(t,y) = C$ is an implicit solution, then

$$\boxed{3y - \frac{y}{t} + \frac{1}{2}y^2 = C}$$

is the implicit sol'n of the Diff. Eq.

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(2) We have the I.V.P.

$$\ddot{y} + 2\dot{y} + 5y = 0; \quad y(0) = 0, \quad \dot{y}(0) = -1.$$

The characteristic eq'n is

$$r^2 + 2r + 5 = 0,$$

since the Diff. Eq'n is linear, homogeneous and const coeff's. (so $y(t) = e^{rt}$).

The solutions to the characteristic eq'n are.

$$r_{1,2} = \frac{-2 \pm \sqrt{4 - 4\cdot 1\cdot 5}}{2} = \frac{-2 \pm 2\sqrt{1-5}}{2}$$

$$= -1 \pm \sqrt{-4} \Rightarrow \boxed{r_{1,2} = -1 \pm 2i}$$

Then, the general sol'n is:

$$y(t) = C_1 (\cos 2t) e^{-t} + C_2 (\sin 2t) e^{-t}$$

i.e. $y(t) = e^{-t}(C_1 \cos 2t + C_2 \sin 2t)$.

We determine the constants using the initial conditions:

$$\dot{y}(t) = -e^{-t}\left(C_1 \cos 2t + C_2 \sin 2t\right)$$
$$+2e^{-t}\left(-C_1 \sin 2t + C_2 \cos 2t\right).$$

~~i.e.~~ Now:

$$y(0) = 0$$

i.e.

$$1 \cdot (C_1 \cdot 1 + C_2 \cdot 0) = 0$$
$$C_1 + 0 = 0 \Rightarrow \boxed{C_1 = 0}$$

Hence:

$$\dot{y}(t) = -e^{-t} C_2 \sin(2t) + 2e^{-t} C_2 \cos(2t)$$

Now $\dot{y}(0) = -1$. Then

$$-1 \cdot C_2 \cdot 0 + 2 \cdot 1 \cdot C_2 \cdot 1 = -1$$

i.e.

$$2C_2 = -1 \Rightarrow C_2 = -\frac{1}{2}$$

Therefore, the sol'n to the I.V.P. is:

$$\boxed{y(t) = -\frac{1}{2} e^{-t} \sin(2t)}$$

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(3) We have to solve the Diff. Eq'n:

$$\ddot{y} + 4\dot{y} + 4y = 8e^{-2t}$$

We require to solve the homogeneous eq'n first,

$$\ddot{y}_h + 4\dot{y}_h + 4y_h = 0$$

which is also linear and Const. Coeff's.

Then, $y_h(t) = e^{rt}$, and so:

$$r^2 + 4r + 4 = 0 \Rightarrow (r+2)^2 = 0$$

We have repeated solutions

$$\boxed{r_1 = r_2 = -2.}$$

Hence:

$$y_h(t) = C_1 e^{-2t} + C_2 t e^{-2t}$$

i.e.

$$\boxed{y_h(t) = e^{-2t}(C_1 + C_2 t).}$$

Now, the forcing term is:

$$f(t) = 8e^{-2t}$$

We may propose, as a judicious conjecture:

$$Y_1(t) = \alpha e^{-2t},$$

but this repeats the sol'n to the homogeneous eq'n.

We might also propose:

$$Y_2(t) = \alpha t e^{-2t},$$

but this sol'n also reproduces a sol'n of the homog. eq'n.

Now, propose:

$$y_p(t) = \alpha t^2 e^{-2t},$$

and this should work:

$$\dot{y}_p(t) = 2\alpha t e^{-2t} + (-2)\alpha t^2 e^{-2t}$$

$$= 2\alpha e^{-2t}(t - t^2)$$

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Now:
$$\ddot{y}_p = -4\alpha e^{-2t}(t-t^2) + 2\alpha e^{-2t}(1-2t)$$
$$= 2\alpha e^{-2t}\left(-2(t-t^2) + (1-2t)\right)$$
$$= 2\alpha e^{-2t}\left(2t^2 - 4t + 1\right)$$

Hence, substitute into the Diff. Eq'n:

$$2\alpha e^{-2t}(2t^2-4t+1) + 4\left(2\alpha e^{-2t}(t-t^2)\right) + 4\alpha t^2 e^{-2t} =$$
$$= 8e^{-2t}$$

$$\alpha \cancel{e^{-2t}}\left(\cancel{4t^2} - \cancel{8t} + 2 + \cancel{8t} - \cancel{8t^2} + \cancel{4t^2}\right) = 8\cancel{e^{-2t}}$$

$$\alpha(2) = 8 \Rightarrow \alpha = 4.$$

$$\Rightarrow \boxed{y_p(t) = 4t^2e^{-2t}}$$

and the general solution is:

$$\boxed{y(t) = e^{-2t}\left(C_1 + C_2 t + 4t^2\right).}$$

(4) We have the Diff. Eq'n.

$$\ddot{y} + 4\dot{y} + 4y = 8e^{-2t},$$

which is the same Diff. Eq'n as in Problem 3, so we should get the same solution.

First, we find the sol'n to the homogeneous equation:

$$\ddot{y}_h + 4\dot{y}_h + 4y_h = 0,$$

which is a <u>linear</u>, <u>homogeneous</u> and <u>const. coeff's</u> <u>Diff. Eq'n</u>.
We then propose, $y_h(t) = e^{rt}$, $r = \text{const}$, so we get the characteristic eq'n:

$$r^2 + 4r + 4 = 0$$

this is to say $\quad (r+2)^2 = 0.$

We have repeated roots: $r_1 = r_2 = -2$. Hence:

$y_h(t) = c_1 y_1(t) + c_2 y_2(t)$, with $y_1(t) = e^{-2t}$; $y_2(t) = te^{-2t}$

Its derivatives are $\quad \dot{y}_1(t) = -2e^{-2t}$; $\dot{y}_2(t) = (1-2t)e^{-2t}$,

and its wronskian

$$W[y_1, y_2](t) = \det \begin{pmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & (1-2t)e^{-2t} \end{pmatrix}$$

$$= (1-2t)e^{-4t} - (-2)te^{-4t} = e^{-4t} \neq 0 \text{ always.}$$

i.e.: $W[y_1, y_2](t) = e^{-4t}$.

By the method of Variation of parameters, we get: the particular solution is:

$$y_p(t) = A(t)\, y_1(t) + B(t)\, y_2(t),$$

with:

$$A(t) = -\int \frac{y_2(t)\, f(t)}{a(t)\, W[y_1, y_2](t)}\, dt$$

$$B(t) = \int \frac{y_1(t)\, f(t)}{a(t)\, W[y_1, y_2](t)}\, dt.$$

and $f(t) = 8e^{-2t}$

$a(t) = 1 \qquad W[y_1, y_2](t) = e^{-4t}$

thus:

$$A(t) = -\int \frac{te^{-2t}\; 8e^{-2t}}{1\cdot e^{-4t}}\, dt = -\int 8t\, dt = -4t^2$$

$$B(t) = \int \frac{e^{-2t}\cdot 8e^{-2t}}{1\cdot e^{-4t}} = \int 8\, dt = 8t$$

Then:

$$y_p(t) = (-4t^2)\, e^{-2t} + (8t)\, te^{-2t}$$

$$\Rightarrow \boxed{y_p(t) = 4t^2 e^{-2t}}$$

Then, we get the same solution as in problem 3

(5) Plutas extra.

We have the solution:

$$y(t) = Ce^{-3t} + 3t^2e^{-3t}$$

Then $y_h(t) = Ce^{-3t}$ is a sol'n of homogeneous eq'n

and $y_p(t) = 3t^2e^{-3t}$ is the particular solution

Since we only have one solution to the homogeneous eq'n, the eq'n is first order.

Since $y_h(t) = Ce^{-3t}$, then the eq'n is linear, homogeneous and constant coeff's, and it should be:

$$\frac{dy_h}{dt} + 3y_h = 0$$

Now to get the forcing term: $f(t)$,

$$\frac{dy}{dt} + 3y = f(t),$$

we just have to substitute the particular solution:

$$\frac{d}{dt}y_p + 3y_p = \frac{d}{dt}\left(3t^2e^{-3t}\right) + 3\left(3t^2e^{-3t}\right)$$

$$= \left(6te^{-3t} + \cancel{(-9)t^2e^{-3t}}\right) + \left(\cancel{9t^2e^{-3t}}\right)$$

$$= 6te^{-3t}$$

Then, the Diff. Eq'n is. $\boxed{\frac{dy}{dt} + 3y = 6te^{-3t}}$

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