UNIVERSIDAD AUTÓNOMA METROPOLITANA - AZCAPOTZALCO
ECUACIONES DIFERENCIALES ORDINARIAS
TRIMESTRE: INVIERNO DE 2018.

EXAMEN # 3.
FECHA: **MIÉRCOLES 4 DE ABRIL DE 2018**

Nombre: ANSWER KEY.

Instrucciones:

- El examen consta de **TRES** problemas, de 35, 30 y 35 puntos, respectivamente.
- Tienen **una** hora con **veinte (25)** minutos para resolverlos.
- Por favor **apaguen sus celulares**. Evíten la pena de quitarles sus exámenes.
- Para recibir puntaje, escriba de forma clara y concisa. Entregue su trabajo limpio y con sus ideas en orden. SIMPLIFIQUE. Muestre sus cuentas. **EXPLIQUE, ARGUMENTE y JUSTIFIQUE** sus respuestas.
- Problema **SIN explicación, desarrollo, justificación o argumento** vale **CERO** puntos.

PROBLEMAS

(1) **(35 puntos.)** En un corral de dimensiones enormes y abundante pastura, el 1 de enero de 2015 fueron colocados 1000 conejos (hembras y machos). El 1 de enero de 2018 se contaron 64,000 conejos.

(a) Calcule la tasa de crecimiento.
(b) ¿Cuántos conejos habrá el 1 de enero de 2020?
(c) ¿Cuánto tiempo tarda en duplicarse la población?
(d) ¿Cuándo habrá 256,000 conejos?

(2) **(30 puntos.)** Resuelva la ecuación diferencial

$$\frac{dy}{dt} = y^{1/2}\left(5(t-2) - \frac{y^{1/2}}{t-2}\right).$$

(3) **(35 puntos.)** Una partícula se sujeta a un resorte dentro de un medio viscoso.

- La masa de la partícula es de 1 kg.
- El resorte se estira 10 cm al aplicarle 1/10 Newtons.
- Denote por b (en unidades N-seg/m), la constante de amortiguamiento.
- La partícula se posiciona a 10 cm a la derecha (de la posición de equilibrio) y se le da un impulso de 1 m/seg a la izquierda.
- Suponga que el movimiento es horizontal.

Responda a lo siguiente.

(a) Determine la constante de Hooke.
(b) Escriba la ecuación de movimiento (sin determinar b, aún).
(c) Suponiendo que el movimiento es **críticamente** amortiguado, determine b.
(d) Escriba la ecuación de movimiento con los valores encontrados.
(e) Encuentre la solución general de la ecuación de movimiento.
(f) ¿Cuál es el factor de amortiguamiento?
(g) Resuelva el problema de valores iniciales.
(h) ¿En qué instante la partícula pasa por la posición de equilibrio?

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Examen #3. Invierno 2018 | Miércoles, Abril 2, 2018

ANSWER KEY

(1) Here, we assume Malthus model of population growth.

$$\frac{dP}{dt} = \alpha P; \qquad \left.\begin{array}{l}[P] = 1 \text{ (inhabitantes)}\\ [t] = \text{years.}\end{array}\right\} \Rightarrow [\alpha] = \frac{1}{\text{year}}.$$

where α is the growth rate. Thus, the solution is:

$$P(t) = P(0)\, e^{\alpha t}.$$

We say, $t = 0_{\text{years}}$ corresponds to January 1^{st}, 2015.
$t = 3_{\text{years}}$ corresponds to January 1^{st}, 2018

Then $1000 = P(0) = P(0)e^{0} = K \Rightarrow \boxed{P(0) = 1000 \text{ rabbits}}$

And so: $\boxed{P(t) = 1000\, e^{\alpha t}}$

(a) On January 1^{st}, 2018, ie., at $t = 3$ years:

$$64{,}000 = P(3) = 1000\, e^{\alpha 3} \Rightarrow 64 = e^{3\alpha}$$

$$\Rightarrow e^{3\alpha} = 2^6 \Rightarrow 3\alpha = \log(2^6) = 6\log 2$$

$$\Rightarrow \boxed{\alpha = 2\log 2 \; \boxed{1/\text{year}}} = \log 4 \, 1/\text{year}$$

is the growth rate.

and: $P(t) = 1000\, e^{(\log 4)t} = e^{\log(4^t)}$

$$\boxed{P(t) = 1000 \cdot 4^t}$$

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(b) On January 1^{st}, 2020, $t = 5$ years. Hence.

$$P(5) = 1000 \cdot 4^5 = 1000 \cdot 2^{10} = 1{,}024{,}000 \text{ rabbits}$$

$$\boxed{P(5) = 1{,}024{,}000 \text{ rabbits}}$$

(c) We want the time, T, where: $P(T) = 2P(0)$

i.e. $$P(0)\, e^{\alpha T} = 2P(0)$$

i.e. $$e^{\alpha T} = 2 \Rightarrow \alpha T = \log 2.$$

$$\Rightarrow T = \frac{1}{\alpha} \log 2 \qquad T = \frac{1}{2\log 2} \cdot \log 2 = \frac{1}{2}$$

$$\boxed{T = 1/2 \text{ year}}$$

(d) Notice that: $256 = 2^8$. Then:

Now, τ is the time we want such that:

$$P(\tau) = 2^8 (1000) = 256{,}000.$$

i.e. $$\cancel{P(0)}\, e^{\alpha\tau} = 2^8 \cdot \cancel{(1000)}$$

$$e^{\alpha\tau} = 2^8$$

$$\alpha\tau = \log(2)^8 = 8\log 2$$

$$\tau = \frac{8\log 2}{\alpha} = \frac{8\log 2}{2\log 2}$$

$$\Rightarrow \tau = \frac{8}{2} \text{ years} = 4 \text{ years}$$

$\boxed{\tau = 4 \text{ years}}$ Then, on January 1^{st}, 2019. there are 256,000 rabbits.

These 4 years are obvious..

In Jan 1^{s}, 2018, we have 64,000 rabbits

Since 1/2 years is the doubling time, then

On July 1st, 2018 there will be 128,000 rabbits

and on January 1st, 2019, there will be 256,000 rabbits

NO CALCULATOR NEEDED!

PROBLEM ② Observe that:

$$\frac{dy}{dt} = y^{1/2}\left(5(t-2) - \frac{y^{1/2}}{t-2}\right)$$

$$= 5(t-2)y^{1/2} - \frac{y}{t-2}$$

i.e.

$$\frac{dy}{dt} + \frac{y}{t-2} = 5(t-2)y^{1/2}$$

is a Bernoulli equation. Here, $\boxed{n = \frac{1}{2}}$. Look for solutions

$y(t)$ of the form: $v(t) = y^{\alpha}(t)$

Then, $\frac{dv}{dt} = \alpha y^{\alpha} \frac{dy}{dt} = \alpha y^{\alpha-1}\left(5(t-2)y^{1/2} - \frac{y}{t-2}\right)$

$$= 5\alpha(t-2)y^{\alpha-1+\frac{1}{2}} - \frac{\alpha y^{\alpha}}{t-2}$$

$$= 5\alpha(t-2)y^{\alpha-1/2} - \alpha\frac{v}{t-2}$$

Then: $\frac{dv}{dt} + \frac{\alpha}{t-2}v = 5\alpha(t-2)y^{\alpha-1/2}$

If $\alpha - \frac{1}{2} = 0$, i.e. $\alpha = \frac{1}{2}$, then:

$$\frac{dv}{dt} + \frac{\alpha}{t-2} v = 5\alpha(t-2)$$

is a linear, non-homogeneous 1st order equation.

Since $\alpha = \frac{1}{2}$: $\frac{dv}{dt} + \underbrace{\frac{1}{2(t-2)}}_{} v = \underbrace{\frac{5}{2}(t-2)}_{}$

If $\frac{dv(t)}{dt} + q(t)v = f(t)$,

then, the integrating factor is:

$$\mu(t) = e^{-\int q(t)dt} = e^{-\int \frac{1}{2(t-2)}dt}$$

$$= e^{-\frac{1}{2}\log|t-2|} = e^{\log|t-2|^{1/2}}$$

$$= \sqrt{t-2}$$

We need to integrate:

$$\int \mu(t) f(t) dt = \int \sqrt{(t-2)}\, \frac{5}{2}(t-2)dt = \frac{5}{2}\int (t-2)^{3/2} dt$$

$$= \frac{5}{2} \cdot \frac{2}{5}(t-2)^{5/2} = (t-2)^{5/2}.$$

Then, the general solution is:

$$v(t) = \frac{1}{\mu(t)} \int \mu(t) f(t) dt + \frac{C}{\mu(t)}$$

$$v(t) = \frac{1}{\sqrt{t-2}}(t-2)^{5/2} + \frac{C}{\sqrt{t-2}} = (t-2)^2 + C(t-2)^{-1/2}$$

Since: $v(t) = y^{1/2}(t) \; \left(= y^{\alpha}(t)\right)$ then $y(t) = v^2(t)$

and so:

$$\boxed{y(t) = \left(C(t-2)^{-1/2} + (t-2)^2\right)^2}$$

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(3) (a) Hooke's law says: $\Delta F = k \, \Delta L$.

Here: $\Delta F = \frac{1}{10} N$. $\Delta L = 10 \text{ cm} = \frac{1}{10} m$.

$\Rightarrow \quad \frac{1}{10} \cdot N = k \cdot \frac{1}{10} m \Rightarrow \boxed{k = 1 \cdot N/m}$

(b) Since there are no external forces:

$m\ddot{y} + b\dot{y} + ky = 0$, then $\boxed{\ddot{y} + b\dot{y} + y = 0}$

(c). To have a critically damped oscillator, we require two repeated roots of the characteristic equation:

$$r^2 + br + 1 = 0$$

i.e., we require: $b^2 - 4mk = 0$
$b^2 - 4 = 0 \quad \Rightarrow b = \pm 2$.

Since $b > 0$, $\boxed{b = 2 \cdot \frac{N \, seg}{m}}$

(d) $\ddot{y} + 2\dot{y} + y = 0$.

(e) Since it is a linear, Const. Coeff's, and non-homogeneous equation, we look for solutions

$$y(t) = e^{rt}. \quad - r = \text{const}.$$

i.e.

$$r^2 + 2r + 1 = 0$$

$$(r+1)^2 = 0, \quad \Rightarrow \boxed{r_1 = r_2 = -1/\text{seg}}$$

Repeated roots

Then: $\boxed{y(t) = (C_1 t + C_2)e^{-t}}$

(f) The damping factor is $\boxed{r = -1/\text{seg}.}$

or $\boxed{e^{-t}}$

(g) If we choose a horizontal axis, oriented as follows:

Then; $y(0) = \frac{1}{10}$ m

$\dot{y}(0) = -1$ m/seg.

Now $\frac{1}{10} = y(0) = (0 + C_2)\cdot 1 = C_2$ $\boxed{C_2 = \frac{1}{10}}$

Also: $\dot{y}(t) = -e^{-t}(C_1 t + C_2) + e^{-t}(C_1)$

Hence: $\quad -1 = \dot{y}(0) = -1(0 + C_2) + 1 \cdot C_1.$

$$C_2 - 1 = C_1 \Rightarrow C_1 = -1 + C_2 = -1 + \frac{1}{10}.$$

$$\boxed{C_1 = -\frac{9}{10}}$$

Therefore: $\boxed{y(t) = \left(-\frac{9}{10}t + \frac{1}{10}\right)e^{-t}}$

(h) The particle passes through the origin when $y(t) = 0$, i.e. $\left(-\frac{9}{10}t + \frac{1}{10}\right)e^{-t} = 0$

Since $e^{-t} > 0$, then, we must have:

$$-\frac{9}{10}t + \frac{1}{10} = 0 \Rightarrow \boxed{t = \frac{1}{9}\text{ sec}}.$$