EXAMEN GLOBAL DE **INTRODUCCIÓN AL CÁLCULO** Viernes, Abril 6, 2018.

Trimestre 18 I. Turno vespertino.

Nombre: ANSWER KEY. Matrícula:

El examen global completo consta de los ejercicios que se encuentran marcados con el símbolo • y cada uno tiene un puntaje. El tiempo de duración es de tres horas. Todas las respuestas deben tener su desarrollo.

PRIMERA PARTE

1. Determinar el dominio y las raíces de las siguientes funciones:

$$f(x) = \sqrt{\frac{2x}{x^2 - x}} \qquad g(x) = \frac{|4x - 2| + 3}{5 - x^2}$$

• **2. (15 puntos)** Sea

$$f(x) = \begin{cases} 2 + (1 + x)^2 & si \quad -4 \le x < -1 \\ 2 - |x| & si \quad -1 \le x < 3 \\ 2 - (1 - x)^2 & si \quad 3 < x \end{cases}$$

a) Elaborar el bosquejo de la gráfica de $f(x)$.
b) Determinar el dominio y las raíces (ceros).
c) Determinar la paridad, la monotonía y el rango.
d) Encontrar los intervalos donde $f(x) > 0$ y donde $f(x) \le 0$.

• **3. (10 puntos)** Considerar las siguientes funciones:

$$f(x) = \sqrt{25 - x^2} \quad \text{y} \quad g(x) = \frac{3 + x^2}{x^2 - 4}$$

a) Determinar el dominio y las raíces de cada función.
b) Obtener la expresión para las funciones $\frac{f}{g}$ y $g \circ f$, así como sus respectivos dominios.

• **4. (10 puntos)** Un rectángulo está inscrito en una circunferencia de radio 5. Expresar el área del rectángulo en función de su ancho.

SEGUNDA PARTE

• **1. (15 puntos)** Calcular los siguientes límites:

$$\lim_{x \to 0} \frac{tan^2(2x)}{x\, sen(2x)}, \qquad \lim_{x \to -3} \frac{3 + x}{2 - \sqrt{x^2 - 5}} \quad \text{y}$$

$$\lim_{x \to \infty} \frac{2x^{\left(\frac{2}{3}\right)} + x}{\sqrt[3]{3x^2 + 2x - 1}}$$

2. Sea

$$g(x) = \frac{-x}{\sqrt{25 - x^2}}$$

a) Determinar el dominio, las raíces (ceros) y la paridad de la función.
b) Obtener las ecuaciones de las asíntotas verticales y horizontales.
c) Elaborar un bosquejo de su gráfica.

• **3. (10 puntos)** En el intervalo $[0, 2\pi]$, realizar el bosquejo de la gráfica de:

$$g(x) = -3\,\mathrm{sen}\left(x - \frac{\pi}{4}\right) + 1.$$

Determinar también la amplitud, el periodo y el rango de $g(x)$.

TERCERA PARTE

• **1. (20 puntos)** Sea

$$f(x) = \frac{-x^2 - x + 12}{x^2 - 4x + 3}$$

Determinar:
a) Dominio, raíces (ceros) y paridad
b) Intervalos de continuidad y clasificar sus discontinuidades.
c) Ecuaciones de asíntotas horizontales y verticales.
Además, elaborar un bosquejo de la gráfica de la función.

• **2. (10 puntos)** Obtener los valores de a y b para que la siguiente función sea continua en todo su dominio.

$$f(x) = \begin{cases} b\, sen\, x + a & si \quad x < -\frac{\pi}{2} \\ 1 - cos\, x & si \quad -\frac{\pi}{2} \le x \le \pi \\ a\, sen\, x + b & si \quad x > \pi \end{cases}$$

3. Encontrar un intervalo de longitud $\frac{\pi}{4}$ o menor, que contenga una solución de la ecuación $sen\, x + 2 - x = 0$. Justificar la respuesta.

• **4. (10 puntos)** Usando la definición de derivada, hallar la ecuación de la recta tangente a la gráfica de la función $f(x) = \sqrt{2x + 1}$ en el punto $\left(-\frac{1}{4}, \frac{1}{\sqrt{2}}\right)$.

INTRO TO CALCULUS UAM - AZCAPOTZALCO Trimestre 18-I.

EXAMEN GLOBAL . ANSWER KEY. Viernes, Abril 6, 2018.

PARTE (I) :

(a) We have: $f(x) = \sqrt{\dfrac{2x}{x^2-x}} = \dfrac{\sqrt{2}\sqrt{x}}{\sqrt{x}\sqrt{x-1}}$. Hence, $x>0$. i.e. $x \in (0,\infty)$

Notice that $f(x) = \dfrac{\sqrt{2}}{\sqrt{x-1}}$, if $x \neq 0$, which is true.

Now, from the denominator $x-1>0$. Then $x>1$. i.e $x \in (1,\infty)$

Hence: $\mathrm{Dom}(f) = (0,\infty) \cap (1,\infty)$

Then: $\boxed{\mathrm{Dom}(f) = (1,\infty)}$

Now $f(x) = \dfrac{\sqrt{2}}{\sqrt{x-1}}$ cannot be zero, $(\sqrt{2} \neq 0)$. Then,

$\boxed{\text{there is no zeros (or roots)}}$.

(b) $g(x) = \dfrac{|4x-2|+3}{5-x^2}$

We impose $5-x^2 \neq 0 \Rightarrow (\sqrt{5}-x)(\sqrt{5}+x) \neq 0$

$\Rightarrow \boxed{\begin{matrix} x_1 \neq \sqrt{5} \\ x \neq -\sqrt{5} \end{matrix}}$

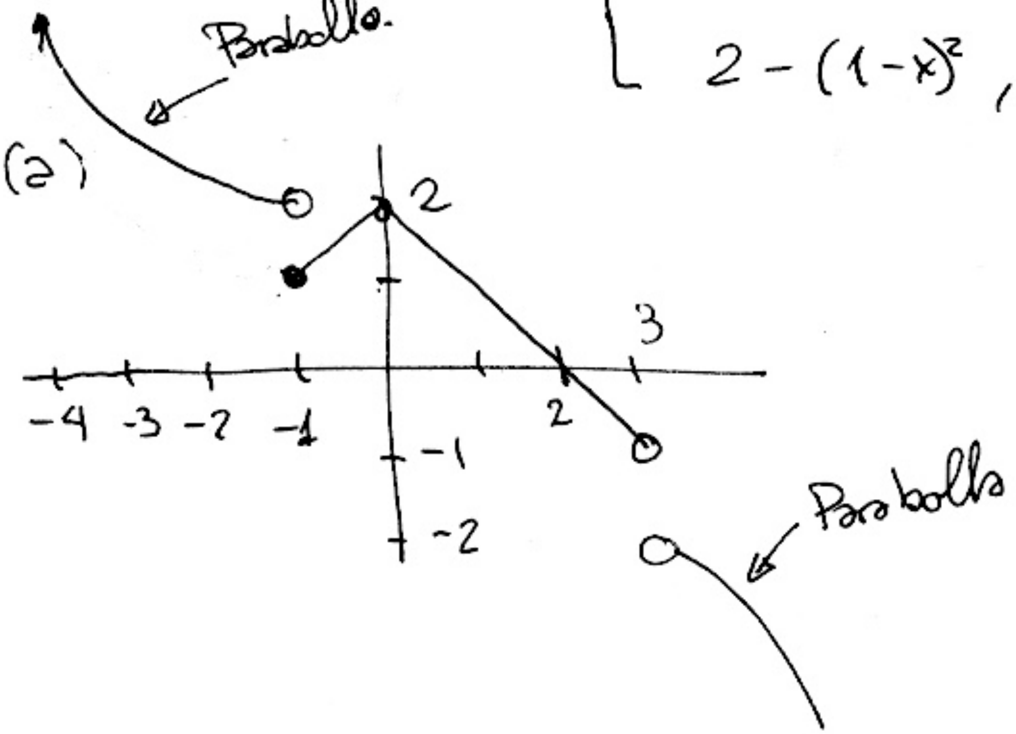
Then, $\boxed{\mathrm{Dom}(g) = \mathbb{R} \setminus \{-\sqrt{5}, \sqrt{5}\}.}$

Now, $\left.\begin{matrix} |4x-2| \geq 0 \\ 3 > 0 \end{matrix}\right\} \Rightarrow |4x-2|+3>0.$

$\boxed{\text{Therefore } g(x) \text{ does not have any zeros or roots}}$

② We have: $f(x) = \begin{cases} 2+(1+x)^2, & -4 \le x < 1 \\ 2-|x|, & -1 \le x < 3 \\ 2-(1-x)^2, & 3 < x \end{cases}$

(a)



(b) Dom(f) $= [-4, 3) \cup (3, \infty)$

The only zero is at $x = 2$

(c) It is neither even or odd.

It is decreasing on the intervals $(-4,-1)$, $(0,3)$ and $(3, \infty)$.

It is increasing on $(-1, 0)$.

Rang(f) $= (-\infty, -2) \cup (-1, 11]$.

(d) $f(x) > 0$ in $[-4, 2)$ and $f \le 0$ in $[2, 3) \cup (3, \infty)$.

③ (a) $f(x) = \sqrt{25 - x^2}$.

We require $25 - x^2 \geq 0$ i.e. $x^2 \leq 25$, i.e. $|x| \leq 5$,

i.e. $-5 \leq x \leq 5. \Longrightarrow \boxed{\text{Dom}(f) = [5, 5]}$

$f(x) = 0$, if $25 - x^2 = 0$, i.e $x^2 - 25 = 0$

$(x-5)(x+5) = 0 \Rightarrow \boxed{\begin{array}{l} x_1 = 5 \\ x_2 = -5 \end{array}}$

For $g(x) = \dfrac{3+x^2}{x^2-4}$. We require $x^2 - 4 \neq 0 \Rightarrow \boxed{\begin{array}{l} x_1 \neq 2 \\ x_2 \neq -2 \end{array}}$

$(x-2)(x+2) \neq 0$

Hence $\boxed{\text{Dom}(g) = \mathbb{R} \setminus \{-2, 2\}}$

Since $\left.\begin{array}{l} 3 > 0 \\ x^2 \geq 0 \end{array}\right\} \Rightarrow 3 + x^2 > 0$, and $g(x)$ can never be zero. $\boxed{g(x) \text{ does not have zeros or roots}}$

Since $g(x) \neq 0$, then $\text{Dom}\left(\frac{f}{g}\right) = \text{Dom}(f) \cap \text{Dom}(g)$.

$$\boxed{\text{Dom}\left(\frac{f}{g}\right) = [-5, -2) \cup (-2, 2) \cup (2, 5]}$$

$\dfrac{f(x)}{g(x)} = \dfrac{\sqrt{25 - x^2}}{\left(\dfrac{3+x^2}{x^2-4}\right)}$ i.e. $\boxed{\dfrac{f(x)}{g(x)} = \dfrac{(x^2-4)\sqrt{25-x^2}}{3+x^2}}$

Now: $(g \circ f)(x) = g(f(x)) = \dfrac{3 + (f(x))^2}{(f(x))^2 - 4} = \dfrac{3 + 25 - x^2}{25 - x^2 - 4}$.

i.e.

$$(g\circ f)(x) = \frac{28-x^2}{21-x^2}.$$

Hence $x \neq \pm\sqrt{21}$.

Now, for $g(f(x))$ to make sense, $x \in \mathrm{Dom}(f)$, i.e. $x \in [-5,5]$, and $f(x) \in \mathrm{Dom}(g)$, i.e. $\mathrm{Ran}(f) \subset \mathrm{Dom}(g)$

or. $\mathrm{Ran}(f) \cap \mathrm{Dom}(g)$.

But $\mathrm{Ran}(f) = [0,5)$.

and. $\mathrm{Dom}(g) = \mathbb{R} \setminus \{-2,2\}$.

Then: $\mathrm{Ran}(f) \cap \mathrm{Dom}(g) = [0,2) \cup (2,5]$.

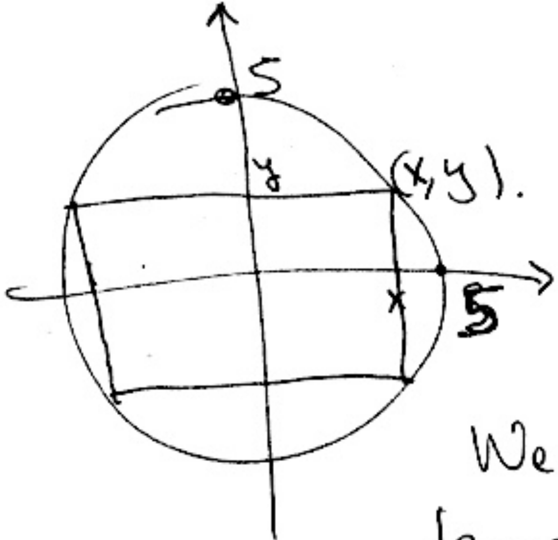
Hence. $f(x) \neq 2$ i.e. $\sqrt{25-x^2} \neq 2$.

$$25 - x^2 \neq 4$$

$$x^2 \neq 21 \Rightarrow x \neq \pm\sqrt{21}.$$

Then: $\boxed{\mathrm{Dom}(g\circ f) = [-5,-\sqrt{21}) \cup (-\sqrt{21},\sqrt{21}) \cup (\sqrt{21},+5).}$

(4)



Since (x,y) is on the circle,

$$x^2+y^2=25.$$

Now, $2x = \text{width} = w$.

We have to find the area in terms of w.

Now: $y = \sqrt{25-x^2}$.

Also: $\text{Area} = (2x)(2y) = 4xy = 4x\sqrt{25-x^2}$.

Put $x = \frac{w}{2}$. Then:

$$\boxed{A(w) = 4\,\frac{w}{2}\sqrt{25-\left(\frac{w}{2}\right)^2}}$$

or $\boxed{A(w) = w\sqrt{100-w^2}}$

PARTE II.

1(a) $\lim_{x\to 0} \frac{\tan^2(2x)}{x\sin(2x)} = \lim_{x\to 0}\left(\frac{1}{\cos^2(2x)} \frac{\sin^2(2x)}{x\sin(2x)}\right) =$

$$= \lim_{x\to 0}\left(\frac{1}{\cos^2(2x)} \frac{\sin 2x}{x}\right)$$

$$= \left(\lim_{x\to 0} \frac{1}{\cos^2(2x)}\right)\left(2\lim_{x\to 0} \frac{\sin(2x)}{2x}\right)$$

$$= 1\cdot 2\cdot 1 = \boxed{2}$$

(b) $\lim_{x\to -3} \frac{3+x}{2-\sqrt{x^2-5}}$, direct substitution does not work

since $3+x = 3+(-3) = 0$

and $2-\sqrt{x^2-5} = 2-\sqrt{(-3)^2-5}$

$= 2-\sqrt{9-5}$

$= 2-\sqrt{4} = 0$

Hence, we should simplify:

$$\lim_{x\to -3} \frac{3+x}{2-\sqrt{x^2-5}} = \lim_{x\to -3} \frac{(3+x)(2+\sqrt{x^2-5})}{(2-\sqrt{x^2-5})(2+\sqrt{x^2-5})}$$

$$= \lim_{x\to -3} \frac{(3+x)(2+\sqrt{x^2-5})}{(2^2-(x^2-5))}$$

$$= \lim_{x\to -3} \frac{(3+x)(2+\sqrt{x^2-5})}{9-x^2}$$

$$= \lim_{x\to -3} \frac{2+\sqrt{x^2-5}}{3-x} = \frac{2+\sqrt{9-5}}{6} = \boxed{\frac{2}{3}}$$

$=6=$

(c) $\displaystyle \lim_{x\to\infty} \frac{2x^{2/3}+x}{\sqrt[3]{3x^2+2x-1}} = \lim_{x\to\infty} \frac{x\left(\frac{2}{x^{1/3}}+1\right)}{x^{2/3}\sqrt[3]{3+\frac{2}{x}-\frac{1}{x^2}}}$

$$= \left(\lim_{x\to\infty} x^{1/2}\right)\left(\lim_{x\to\infty} \frac{2/x^{1/3}+1}{\sqrt[3]{3+\frac{2}{x}-\frac{1}{x^2}}}\right)$$

$$= \left(\lim_{x\to\infty} x^{1/2}\right)\frac{1}{\sqrt[3]{3}} = \boxed{\infty}$$

2(a). $g(x) = \dfrac{-x}{\sqrt{25-x^2}}$

(a) For Dom(g), we require $25-x^2>0$, $x^2<25$, $|x|<25$, $-5<x<5$ $\Rightarrow$ $\boxed{\mathrm{Dom}(g)=(-5,5)}$

Zeros (Roots) $g(x)=0$, $\dfrac{-x}{\sqrt{25-x^2}}=0 \Rightarrow \boxed{x=0}$ is the only zero

Parity
It is an odd function: $g(-x) = \dfrac{-(-x)}{\sqrt{25-(-x)^2}} = -\left(\dfrac{-x}{\sqrt{25-x^2}}\right)$

$= -g(x)$, it is odd.

(b) When $x\to5^-$, $\displaystyle \lim_{x\to5^-}\frac{-x}{\sqrt{25-x^2}} = -5\lim_{x\to5^-}\left(\frac{1}{\sqrt{25-x^2}}\right) = \boxed{-\infty}$

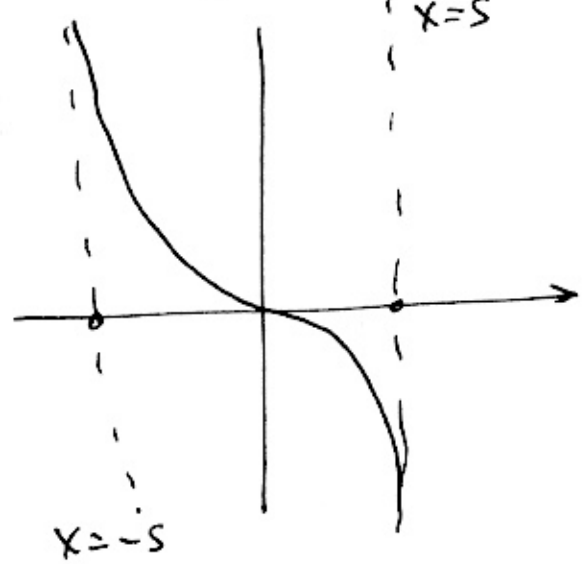
When $x\to-5^+$, $\displaystyle \lim_{x\to-5^+}\left(\frac{-x}{\sqrt{25-x^2}}\right) = 5\lim_{x\to-5^+}\frac{1}{\sqrt{25-x^2}} = \boxed{+\infty}$

Then, $\boxed{\begin{matrix} x=5 \\ x=-5 \end{matrix}}$ are the vertical asymptotes.

Since $\mathrm{Dom}(g) = (-5,5)$, we cannot take $x \longrightarrow \infty$.

Then, there is no horizontal asymptotes.

The graph looks like:



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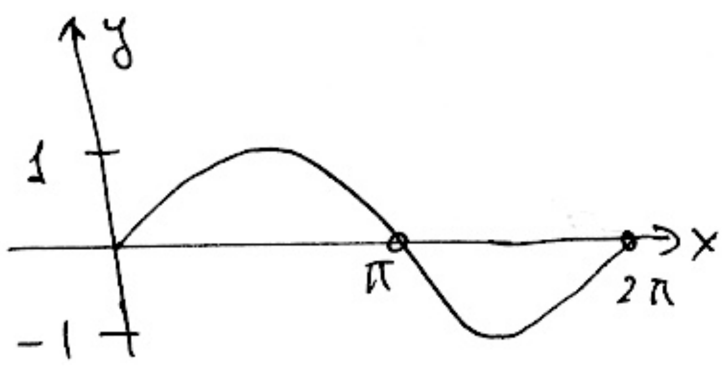
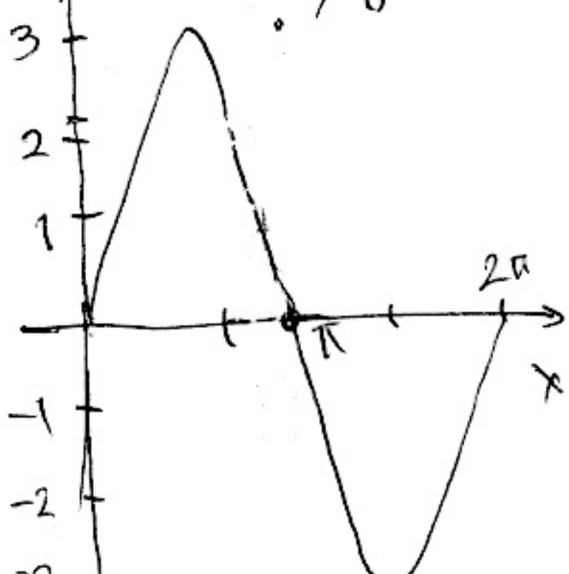
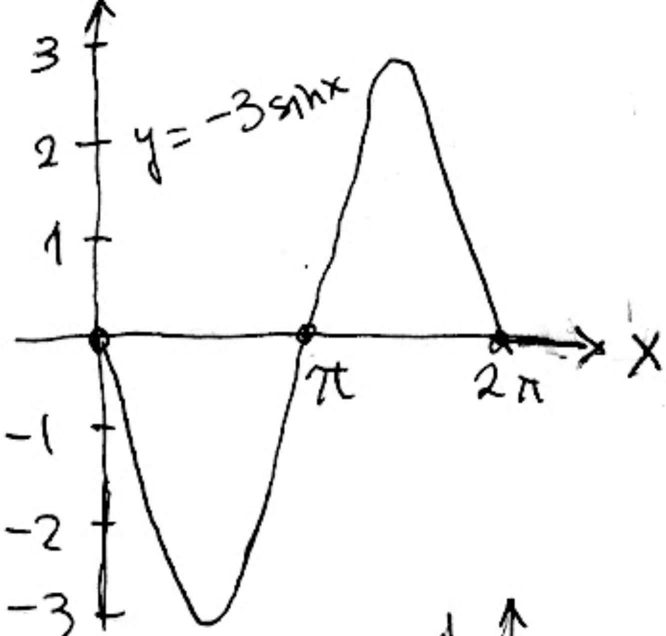
(3) We start with

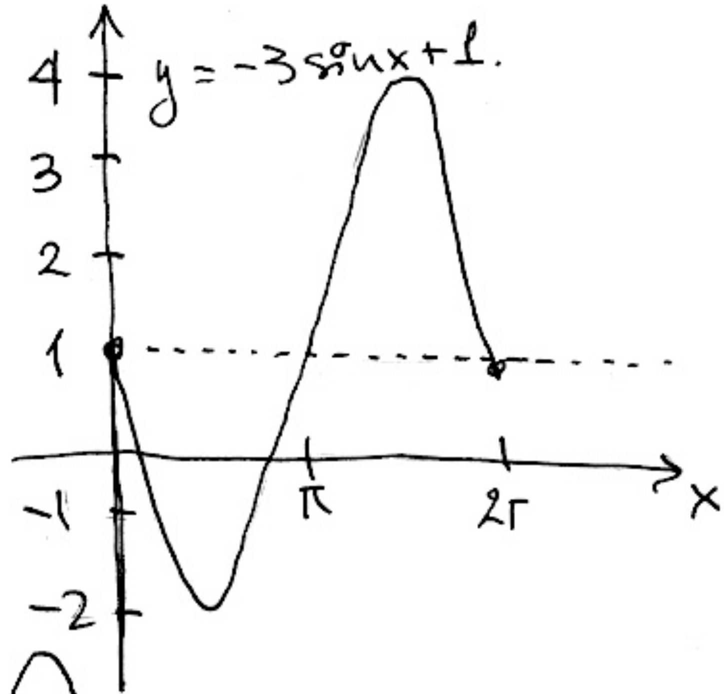
$y = \sin x$



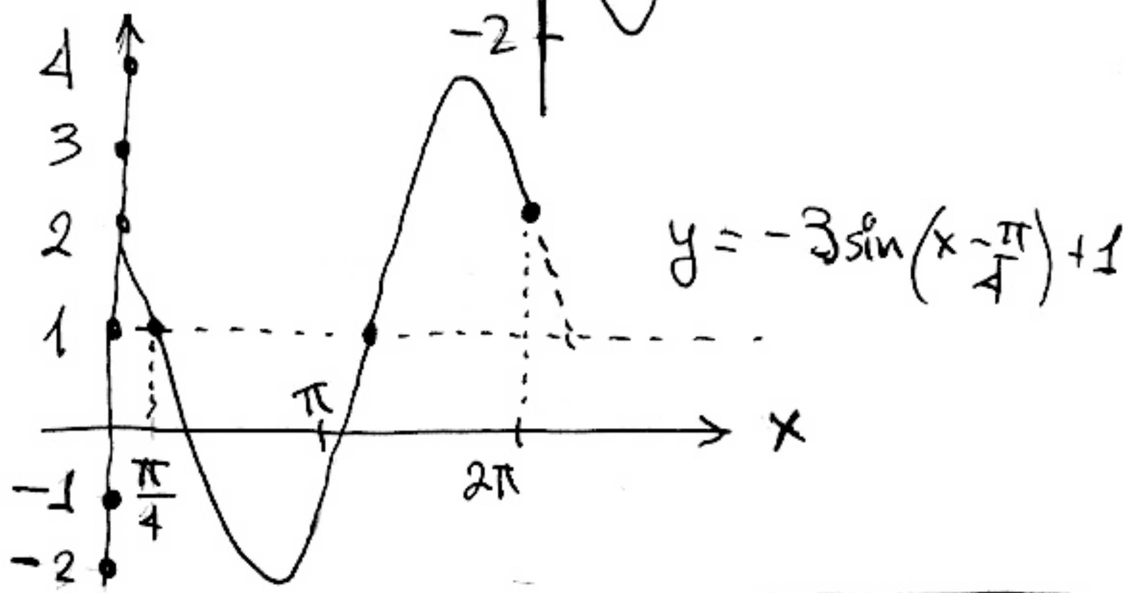
Now, $y = 3\sin x$.







Finally,



Since we do <u>not</u> multiply x, the $\boxed{\text{period is } 2\pi.}$

The amplitude is 3

and $\mathrm{Rang}(f) = [-2, 4]$.

PARTE III.

(1) $f(x) = \dfrac{-(x^2+x-12)}{x^2-4x+3} = -\dfrac{(x+4)(x-3)}{(x-3)(x-1)}$.

(a) Dom(f) = $\mathbb{R}\setminus\{1, 3\}$

Since $x \neq 3$, $x-3 \neq 0$ and so:

$$f(x) = -\frac{(x+4)}{x-1}$$

Roots $f(x)=0$, if $\boxed{x=-4}$ and it is the only root.

Parity f is neither even nor odd.

(b) It is continuous on the intervals $(-\infty,1)$, $(1,3)$ and $(3,\infty)$.

The point $x=1$ is an infinite discontinuity.

The point $x=3$ is a removable discontinuity.

(c) Now

$$\lim_{x\to 1^+} -\frac{(x+4)}{x-1} = -5 \lim_{x\to 1^+} \frac{1}{x-1} = (-5)\infty = -\infty$$

and $$\lim_{x\to 1^-} -\frac{(x+4)}{x-1} = (-5)\lim_{x\to 1^-} \frac{1}{x-1} = (-5)(-\infty) = +\infty$$

Then $x=1$ is an infinite discontinuity

and $x=1$ is a vertical asymptote.

Now $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} -\dfrac{(x+4)}{x-1} = (-1)\lim_{x\to\infty} \dfrac{x(1+4/x)}{x(1-1/x)}$

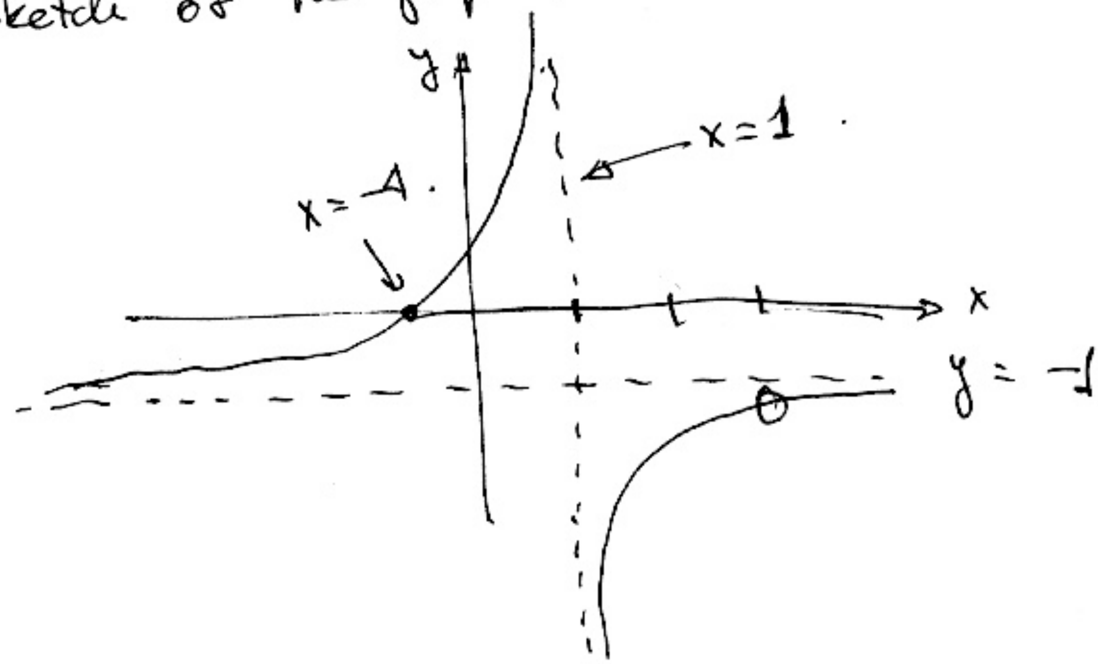
$$= (-1) \lim_{x \to \infty} \left(\frac{1 + 4/x}{1 - 1/x} \right) = (-1)(1) = 1$$

Similarly for $x \to -\infty$

$$\lim_{x \to -\infty} f(x) = (-1) \lim_{x \to -\infty} \frac{1 + 4/x}{1 - 1/x} = (-1)(1) = 1$$

Then $\boxed{y = 1}$ is a horizontal asymptote.

The sketch of the graph looks like:



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(2) We have the function.

$$f(x) = \begin{cases} b \sin x + a, & x < -\frac{\pi}{2} \\ 1 - \cos x, & -\frac{\pi}{2} \le x \le \pi \\ a \sin x + b, & x > \pi. \end{cases}$$

The points of potential discontinuities are $x = -\frac{\pi}{2}$, $x = \pi$.

Since $\sin x$ and $\cos x$ are continuous function, then $f(x)$ is continuous on $(-\infty, -\frac{\pi}{2}) \cup (-\frac{\pi}{2}, \pi) \cup (\pi, \infty)$.

Then, we should compute the side-limits at $x = -\frac{\pi}{2}$ and $x = \pi$.

$*\ \lim_{x \to -\frac{\pi}{2}^-} f(x) = \lim_{x \to -\frac{\pi}{2}} (b \sin x + a) = -b + a$

and

$*\ \lim_{x \to -\frac{\pi}{2}^+} f(x) = \lim_{x \to -\frac{\pi}{2}} 1 - \cos x = 1 - 0 = 1$

For $\lim_{x \to -\frac{\pi}{2}} f(x)$ to exist, these two limits should be the same: $\boxed{a - b = 1}$

Also:

$*\ \lim_{x \to \pi^-} f(x) = \lim_{x \to \pi} (1 - \cos x) = 1 - (-1) = 2$

and

$*\ \lim_{x \to \pi^+} f(x) = \lim_{x \to \pi} (a \sin x + b) = a \cdot 0 + b = b$

Then, for $\lim_{x \to \pi} f(x)$ to exist, these limits should be the same: $\boxed{b = 2}$

Then a: $a = b + 1 = 2 + 1 \Rightarrow \boxed{a = 3}$.

Then $\boxed{\lim_{x \to -\frac{\pi}{2}} f(x) = 1}$ and $\boxed{\lim_{x \to \pi} f(x) = 2}$

Now: $f(-\frac{\pi}{2}) = 1 - \cos(-\frac{\pi}{2}) = 1$.

and $f(\pi) = 1 - \cos\pi = 1 - (-1) = 2$.

Hence:

$$\lim_{x \to -\frac{\pi}{2}} f(x) = 1 = f\left(-\frac{\pi}{2}\right)$$

and $\lim_{x \to \pi} f(x) = 2 = f(\pi)$.

Then $f(x)$ is continuous on $x = -\frac{\pi}{2}$ and $x = \pi$

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(3) Consider the function

$$f(x) = \sin x + 2 - x.$$

Notice that:

$$f(0) = 0 + 2 - 0 = 2$$

$$f\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) + 2 - \frac{\pi}{4} = \frac{\sqrt{2}}{2} + 2 - \frac{\pi}{4} \approx 1.92$$

$$f\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) + 2 - \frac{\pi}{2} = 1 + 2 - \frac{\pi}{2} \approx 1.43$$

$$f\left(\frac{3\pi}{4}\right) = \sin\left(\frac{3\pi}{4}\right) + 2 - \frac{3\pi}{4} = \frac{\sqrt{2}}{2} + 2 - \frac{3\pi}{4} \approx 0.35$$

$$f(\pi) = \sin(\pi) + 2 - \pi = 0 + 2 - \pi \approx -1.14.$$

We have $f\left(\frac{3\pi}{4}\right) > 0$ and $f(\pi) < 0$:

i.e. $f(\pi) < 0 < f\left(\frac{3\pi}{4}\right)$.

Since f is continuous ($\sin x$ and $2-x$ are continuous).

then, there is a $c \in \left(\frac{3\pi}{4}, \pi\right)$, such that:

$f(c) = 0$, i.e. c is a root of eq'n $\sin x + 2 - x = 0$

(4) We need to compute:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{2(x+h)+1}-\sqrt{2x+1}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{2(x+h)+1}-\sqrt{2x+1}}{h} \cdot \left(\frac{\sqrt{2(x+h)+1}+\sqrt{2x+1}}{\sqrt{2(x+h)+1}+\sqrt{2x+1}}\right)$$

$$= \lim_{h\to 0} \frac{\left(2(x+h)+1\right)-(2x+1)}{h\left(\sqrt{2(x+h)+1}+\sqrt{2x+1}\right)}$$

$$= \lim_{h\to 0} \frac{2h}{h\left(\sqrt{2(x+h)+1}+\sqrt{2x+1}\right)}$$

$$= \lim_{h\to 0} \frac{2}{\sqrt{2(x+h)+1}+\sqrt{2x+1}} = \frac{2}{\sqrt{2x+1}+\sqrt{2x+1}}$$

i.e.

$$\boxed{f'(x) = \frac{1}{\sqrt{2x+1}}}$$

Evaluating at $x = -\frac{1}{4}$, we have the slope of the tangent line:

$$m = f'\left(-\frac{1}{4}\right) = \frac{1}{\sqrt{2\left(-\frac{1}{4}\right)+1}} = \sqrt{2}.$$

The given points is $\left(-\frac{1}{4}, f\left(-\frac{1}{4}\right)\right)$.

But $f\left(-\frac{1}{4}\right) = \sqrt{2\left(-\frac{1}{4}\right)+1} = \frac{1}{\sqrt{2}}$ i.e., $\left(-\frac{1}{4}, \frac{1}{\sqrt{2}}\right) = (x_0, y_0)$

The equation of a line is $y = m(x-x_0)+y_0$.

Hence the tangent line is:

$$\boxed{y = \sqrt{2}\left(x+\frac{1}{4}\right) + \frac{1}{\sqrt{2}}}$$

or

$$\boxed{y = \sqrt{2}x + \frac{3}{4}\sqrt{2}}$$