UNIVERSIDAD AUTÓNOMA METROPOLITANA
EXAMEN GLOBAL DE ECUACIONES DIFERENCIALES
Trimestre 18I. Turno vespertino. Abril 11 de 2018.

Alumno: Matrícula: Calificación:

El examen global consta de los problemas marcados con (*). Quien presente una de las partes, deberá resolver todos los problemas correspondientes a esa parte. Los resultados deberán mostrar el procedimiento respectivo. **Todos los problemas del examen global tienen el mismo valor.**

PRIMERA PARTE

(1*) Resolver el siguiente problema de valores iniciales

$$(y^2x^2 - e^{2x} + y^2e^{2x} - x^2)dy = (xy^2 + e^{2x} + x + y^2e^{2x})dx, \quad y(0) = 1$$

(2*) Resolver: $(ye^x - y^2 \cos x)\,dx + (2e^x - 3y \operatorname{sen} x)\,dy = 0$

(3*) Resolver la ecuación: $3(1 + x^2)y' = 2xy(y^3 - 1)$

(4*) Obtener la ecuación de las trayectorias ortogonales a la familia de curvas siguiente:

$$y = \frac{c_1}{1 + x^2}.$$

De las trayectorias encontradas, encuentre la que pasa por el punto (1,0).

SEGUNDA PARTE

(5) Resolver el siguiente problema de valores iniciales:

$$y'' + 4y' + 5y = 0, \quad y(0) = 0, \; y'(0) = 7$$

(6*) Dada la ecuación diferencial siguiente

$$\cancel{x}y'' - (x + 1)\,y' + y = 0$$

y una **solución** $y_1(x) = e^x$, encontrar su **solución general**.

(7*). Resolver la siguiente ecuación: $y'' + 6y' + 8y = 3e^{-2x} + 2x$

(8*) Resolver por el **método de variación de parámetros:** ¡Aquí es obvio!

$$y'' + 3y' + 2y = \frac{1}{1 + e^x}$$

TERCERA PARTE

(9*) Un cuerpo que pesa 2 kg se sujeta a un resorte, deformándolo 10 cm. Si el cuerpo se pone en movimiento 5 cm arriba de su posición de equilibrio, con una velocidad de 1 m/s, hacia arriba.

a) Plantear y resolver el problema de valores iniciales.

b) Expresar la solución en su forma alternativa.

c) Obtener la frecuencia, la amplitud y el período del movimiento, así como la cantidad de ciclos completos realizados en 5π segundos.

(10*) Después que un cuerpo que pesa 10 lb se sujeta a un resorte de 5 pies de largo, el resorte mide 7 pies. Se quita el cuerpo de 10 lb y se le reemplaza por uno de 8 lb; el sistema completo se coloca en un medio que ofrece una resistencia numéricamente igual a la velocidad instantánea. Si el cuerpo se suelta desde un punto que está 1/2 pie abajo de la posición de equilibrio con una velocidad dirigida hacia abajo de 1 pie/s,

a) Plantear y resolver el problema de valores iniciales.

b) Determinar la posición del cuerpo a los 10 s de iniciado el movimiento.

(11*) A un resorte que pende del techo se le sujeta un cuerpo de 1 kg del estremo inferior, estirándolo 0.098 m. Si el movimiento se inicia desde el reposo, a 0.1 m debajo de la posición de equilibrio y se aplica una fuerza externa dada por $f(t) = 2\,\text{sen}(10t)$,

a) Plantear y resolver el problema de valores iniciales.

b) Determinar la posición del cuerpo a los 20 s de iniciado el movimiento. Indicar si se presenta el fenómeno de resonancia.

Ecuaciones Diferenciales Ordinarias. Miércoles 11 de abril de 2018

UAM-Azcapotzalco. Examen Global Trimestre Invierno 2015

Soluciones

(1) We have the following Diff. Eq'n:

$$\left(y^2x^2 - e^{2x} + y^2e^{2x} - x^2\right)\frac{dy}{dx} = \left(xy^2 + e^{2x} + x + y^2e^{2x}\right)$$

Then Observe that:

$$\left[y^2(x^2+e^{2x}) - (e^{2x}+x^2)\right]\frac{dy}{dx} = \left[(x+e^{2x})y^2 + (x+e^{2x})\right]$$

$$(y^2-1)(x^2+e^{2x})\frac{dy}{dx} = (x+e^{2x})(y^2+1).$$

Hence, this is a separable eq'n:

$$\int\left(\frac{y^2-1}{y^2+1}\right)dy = \int\left(\frac{x+e^{2x}}{x^2+e^{2x}}\right)dx$$

The first integral is:

$$\int\frac{y^2-1}{y^2+1}\,dy = \int\frac{1+y^2-2}{1+y^2}\,dy = \int\left(1-\frac{2}{1+y^2}\right)dy =$$

$$= y - 2\,\mathrm{Arctan}(y) + C_1$$

The second integral:

(Or using the change of variables $u = x^2+e^{2x}$)

$$\int\frac{x+e^{2x}}{x^2+e^{2x}}\,dx = \frac{1}{2}\int\frac{2x+2e^{2x}}{x^2+e^{2x}}\,dy = \frac{1}{2}\log|x^2+e^{2x}| + C_2.$$

Hence:

$$\boxed{y - 2\,\mathrm{Arctan}(y) = \frac{1}{2}\mathrm{Log}|x^2+e^{2x}| + C}$$

Now, $y(0) = 1$:

$$1 - 2\,\mathrm{Arctan}(1) = \frac{1}{2}\log|0+1| + C$$

$$1 - 2\frac{\pi}{4} = C \qquad C = 1 - \frac{\pi}{2}.$$

Hence we have an explicit solution:

$$\boxed{y(x) - 2\,\mathrm{Arctan}(y(x)) = \frac{1}{2}\log|x^2 + e^{2x}| + 1 - \frac{\pi}{2}}$$

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(2) We have the Diff. Eq:

$$(ye^x - y^2\cos x) + (2e^x - 3y\sin x)\frac{dy}{dx} = 0$$

Let $M(x,y) = ye^x - y^2\cos x$, $N(x,y) = (2e^x - 3y\sin x)$

Computing

$$\frac{\partial M}{\partial y} = e^x - 2y\cos x, \quad \frac{\partial N}{\partial x} = 2e^x - 3y\cos x,$$

we observe that the Diff. Eq. is <u>not</u> exact.

Multiplying by an integrating factor: $\mu(x,y)$:

$$(\mu M) + (\mu N)\frac{dy}{dx} = 0,$$

this Diff Eq'n should be exact, so that:

$$\partial_y(\mu M) = \partial_x(\mu N) \text{ holds.}$$

thus:

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x.$$

Assuming $\mu_y = 0$, we have:

$$\mu_x = \frac{(M_y - N_x)\mu}{N}$$

i.e.

$$\mu_x = \left[\frac{(e^x - 2y\cos x) - (2e^x - 3y\cos x)}{2e^x - 3y\sin x}\right]\mu = \left[\frac{-e^x + y\cos x}{2e^x - 3y\sin x}\right]\mu$$

Hence, $\mu_y \neq 0$.

Assume $\mu_x = 0$. Then:

$$\mu_y = \frac{N_x - M_y}{M}\mu$$

i.e.

$$\mu_y = \left(\frac{e^x - y\cos x}{ye^x - y^2\cos x}\right)\mu = \left(\frac{e^x - y\cos x}{y(e^x - y\cos x)}\right)\mu = \frac{1}{y}\mu$$

Then $\mu_y = \frac{1}{y}\mu \Rightarrow \boxed{\mu(x,y) = y.}$

Hence, the Diff. Eq'n:

$$(y^2e^x - y^3\cos x) + (2ye^x - 3y^2\sin x)\frac{dy}{dx} = 0$$

is now exact:

$$\frac{\partial}{\partial y}(y^2e^x - y^3\cos x) = 2ye^x - 3y^2\cos x = \frac{\partial}{\partial x}(2ye^x - 3y^2\sin x).$$

Hence, there is a function $\Psi(x,y)$, such that

$$\left.\begin{aligned} \frac{\partial\Psi}{\partial x} &= y^2e^x - y^3\cos x \\ \text{and} \quad \frac{\partial\Psi}{\partial y} &= 2ye^x - 3y^2\sin x \end{aligned}\right\} \Rightarrow \begin{aligned} \Psi(x,y) &= y^2e^x - y^3\sin x + f(y) \\ \Psi(x,y) &= y^2e^x - y^3\sin x + g(x), \end{aligned}$$

Hence, $f(y) = g(x) = C$. Therefore, $\boxed{y^2e^x - y^3\sin x = C}$

is an implicit solution.

③ Expanding $3(1+x^2)y' = 2xy^4 - 2xy$,
and rewriting the Diff. Eq'n; we obtain a Bernoulli Diff. Eq'n:

$$\frac{dy}{dx} + \frac{2x}{3(1+x^2)}y = \frac{2x}{3(1+x^2)}y^4$$

Here, we define $v(x) = y^{\alpha}(x)$, α – to be determined:

$$\frac{dv}{dx} = \alpha y^{\alpha-1}\frac{dy}{dx} = \alpha y^{\alpha-1}\left(-\frac{2x}{3(1+x^2)}y + \frac{2x}{3(1+x^2)}y^4\right)$$

$$= \alpha\left(\frac{-2x}{3(1+x^2)}\right)y^{\alpha} + \alpha\frac{2x}{3(1+x^2)}y^{\alpha+3} \underset{\alpha=-3}{=} \frac{-2\alpha x}{3(1+x^2)}v + \frac{2\alpha x}{3(1+x^2)}$$

Hence: $v(x) = y^{-3}(x)$ and:

$$\frac{dv}{dx} - \frac{2x}{1+x^2}v = \frac{-2x}{1+x^2}.$$

The integrating factor is $\mu = e^{-\int\frac{2x}{1+x^2}dx} = e^{-\log(1+x^2)}$

$$= \frac{1}{1+x^2}.$$

Hence,

$$\frac{1}{1+x^2}\frac{dv}{dx} - \frac{2x}{(1+x^2)^2}v = \frac{-2x}{(1+x^2)^2} \quad \text{is exact.}$$

i.e.

$$\frac{d}{dx}\left(\frac{v}{1+x^2}\right) = \frac{d}{dx}\left(\frac{1}{1+x^2}\right)$$

Hence.

$$\frac{v(x)}{1+x^2} = \frac{1}{1+x^2} + \tilde{C} \Rightarrow v(x) = 1 + \tilde{C}(1+x^2)$$

$$\boxed{v(x) = C + \tilde{C}x^2}$$

Hence $y^{-3} = v \Rightarrow y = v^{-1/3}$

$$\boxed{y(x) = \frac{1}{\sqrt[3]{1+\tilde{C}(1+x^2)}}}$$

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(4) To get the orthogonal trajectories of the curves:

$$y(x) = \frac{C_1}{1+x^2}, \quad y(\;) = \;; \qquad (*).$$

The derivative of $y(x)$ at x_0 is the slope m of the tangent line:

$$m = \frac{dy}{dx}(x_0) = \frac{-2C_1 x_0}{(1+x_0^2)^2}$$

Now, from $(*)$: $C_1 = y_0(1+x_0^2)$. Hence:

$$m = \frac{dy}{dx}(x_0) = -\frac{2\left(y_0(1+x_0^2)\right)x_0}{(1+x_0^2)^2} = -\frac{2x_0 y_0}{1+x_0^2}.$$

Then, the slope of the perpendicular line is: $M = \frac{-1}{m}$.

i.e.

$$M = \frac{1}{2x_0 y_0}. \quad \left(= -\frac{1}{m} = \frac{-(1+x_0^2)}{-2x_0 y_0} \right).$$

Now, the slope of the perpendicular curve, $Y = Y(x)$, at $x = x_0$ is: M. thus:

$$\frac{dY}{dx}(x_0) = \frac{1+x_0^2}{2x_0 y_0}.$$

But: $(x_0, y(x_0)) = (x_0, Y(x_0))$, since these two curves pass at the same point. thus $y(x_0) = Y(x_0)$, i.e. $y = Y$, and so, we have the Diff Eq'n:

$$\frac{dY}{dx} = \frac{1+x^2}{2xY}$$

which is separable:

$$\int Y\, dY = \frac{1}{2}\int \left(\frac{1+x^2}{x}\right) dx.$$

$$\frac{1}{2}Y^2 = \frac{1}{2}\left(\mathrm{Log}|x| + \frac{x^2}{2}\right) + C.$$

i.e. $\boxed{Y^2 = \mathrm{Log}|x| + \frac{x^2}{2} + C}$ is the required family of curves.

Now, $Y(1) = 0$:

$$0 = \mathrm{Log}|1| + \frac{1}{2} + C \Leftrightarrow \boxed{C = -\tfrac{1}{2}}$$

$$\boxed{Y^2(x) = \log|x| + \frac{x^2}{2} - \frac{1}{2}} \text{ for } |x| \geq 1$$

$$\mathrm{Dom}(Y) = \{x \in \mathbb{R} / |x| \geq 1\}$$
$$= (-\infty, -1] \cup [1, \infty)$$

SEGUNDA PARTE

(5) Solve the Initial Value Problem.

$$\ddot{y} + 4\dot{y} + 5y = 0, \quad y(0) = 0, \quad y'(0) = 7$$

The Diff. Eq is linear, const. coeff's, homogeneous:

$$y(t) = e^{rt}, \quad r - \text{const to be found.}$$

Then:

$$r^2 + 4r + 5 = 0 \Rightarrow (r \quad)(r \quad) = 0$$

$$r = \frac{-4 \pm \sqrt{16 - 4 \cdot 5}}{2} = \frac{-4 \pm 2\sqrt{4-5}}{2}$$

$$= -2 \pm i.$$

Then:

$$\boxed{y(t) = (C_1 \cos t + C_2 \sin t) e^{-2t}}$$

Now:

$$\dot{y}(t) = (-C_1 \sin t + C_2 \cos t) e^{-2t} + (-2)(C_1 \cos t + C_2 \sin t) e^{-2t}$$

At $t = 0$:

$$y(0) = C_1 = 0.$$

$$\dot{y}(0) = C_2 - 2C_1 = C_2 = 7:$$

Then:

$$\boxed{y(t) = 7 \sin t \, e^{-2t}}$$

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(6) This problem is ill-posed.

$$\ddot{y} - (x+1)\dot{y} + y = 0.$$

However, the solution key sheet shows.

$$x\ddot{y} - (x+1)\dot{y} + y = 0.$$

The Diff Eq'n. $x\ddot{y} - (x+1)\dot{y} + y = 0$

Here, $y_1(x) = e^x$, solves the eq'n.

$$x e^x - (x+1)e^x + e^x = 0 \checkmark$$

Now, look for $v(x)$, such that

$y_2(x) = v(x)y_1(x)$ also is a solution

Hence:

$$\dot{y}_2(x) = \dot{v}\, y_1 + v\dot{y}_1$$

$$\ddot{y}_2 = \ddot{v}\, y_1 + 2\dot{v}\,\dot{y}_1 + v\ddot{y}_1.$$

From this,

$$x\ddot{y}_2 - (x+1)\dot{y}_2 + y_2 = 0$$

implies:

$$x(\ddot{v}y_1 + 2\dot{v}\dot{y}_1 + v\ddot{y}_1) - (x+1)(\dot{v}y_1 + v\dot{y}_1) + vy_1 = 0$$

i.e.

$$(xy_1)\ddot{v} + (2x\dot{y}_1 - (x+1)y_1)\dot{v} + \underbrace{(x\ddot{y}_1 - (x+1)\dot{y}_1 + y_1)}_{=0}v = 0$$

Let $V = \dot{v}$. then.

$$xy_1\dot{V} + (2x\dot{y}_1 - (1+x)y_1)V = 0$$

Since $y_1(x) = e^x$:

$$x\dot{V} + (2x - (1+x))V = 0$$

$$x\dot{V} + (x-1)V = 0 \Rightarrow \frac{\dot{V}}{V} = \frac{1-x}{x}.$$

$$\Rightarrow \frac{d}{dt}(\log V) = \frac{1}{x} - 1 \Rightarrow \log V = \log|x| - x + C_1$$

$$V(x) = C_2 x e^{-x} = C_2 \frac{d}{dx}.$$

$$\dot{v} = C_2 x e^{-x} \Rightarrow v(x) = C_2 \int x e^{-x}\, dx$$

i.e.

$$v(x) = C_2\left[-xe^{-x} + \int e^{-x}\, dx\right] = C_2\left[-xe^{-x} - e^{-x} + C_3\right]$$

$$= -C_2(x+1)e^{-x} + C_2 C_3.$$

$$v(x) = C_4 (1+x)e^{-x} + C_5$$

$$y_2(x) = v(x)\, y_1(x) = v(x)\, e^{x}.$$

$$y_2(x) = C_4(1+x) + C_5 e^{x}.$$

Choos $C_4 = 1$. Already fnd.

Then:

$$\boxed{y_2(x) = 1+x}$$

The general solution is then, by linearity:

$$\boxed{y(x) = K_1 e^{x} + K_2(1+x)}$$

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(7) Solve the Diff Eq'n. $\ddot{y} + 6\dot{y} + 8y = 3e^{-2t} + 2t$:

This is a linear, const. coeff's, non-homogeneous Diff. Eq'n. We use Variation of Parameters

We solve first the homogeneous Diff. Eq'n:

$$\ddot{y}_h + 6\dot{y}_h + 8y_h = 0 \Rightarrow y_h(t) = e^{rt}$$

$$\Rightarrow \quad r^2 + 6r + 8 = 0 \Rightarrow r = \frac{-6 \pm \sqrt{36-32}}{2}$$

$$(r+2)(r+4) = 0 \qquad r = \frac{-6 \pm \sqrt{4}}{2} = -3 \pm 1$$

$$r_1 = -2, \; r_2 = -4$$

$$\Rightarrow \boxed{y_h(t) = C_1 e^{-2t} + C_2 e^{-4t}}$$

Since $f(t) = 3e^{-2t} + 2t$

look for particular solutions of the form.

$$y_p^{(1)}(t) = \alpha e^{-2t} + (\beta t + \gamma)$$

But αe^{-2t} repeats a sol'n to the homg. eq'n. Then should work,

$$y_p(t) = \alpha t e^{-2t} + (\beta t + \gamma)$$

Then: $\dot{y}_p(t) = \alpha e^{-2t} - 2\alpha t e^{-2t} + \beta$

$$\ddot{y}_p(t) = -4\alpha e^{-2t} + 4\alpha t e^{-2t}$$

Substitute into the Diff Eq'n:

$$\left(-4\alpha e^{-2t} + \cancel{4\alpha t e^{-2t}}\right) + 6\left(\alpha e^{-2t} - \cancel{2\alpha t e^{-2t}} + \beta\right) + 8\left(\cancel{\alpha t e^{-2t}} + \beta t + \gamma\right)$$

$$= 3e^{-2t} + 2t.$$

$$(-4\alpha + 6\alpha)e^{-2t} + (6\beta + 8\beta t + 8\gamma) = 3e^{-2t} + 2t$$

$$(2\alpha)e^{-2t} + (8\beta t + 8\gamma + 6\beta) = 3e^{-2t} + 2t$$

Then:

$\boxed{\alpha = \frac{3}{2}}$; $\left.\begin{array}{l} 8\beta = 2 \\ 8\gamma + 6\beta = 0 \end{array}\right\} \Rightarrow \boxed{\beta = 1/4}$; $8\gamma + \frac{6}{4} = 0 \Rightarrow \boxed{\gamma = -\frac{3}{16}}$

$$\Rightarrow \quad y_p(t) = \frac{3}{2} t e^{-2t} + \left(\frac{1}{4} t - \frac{3}{16}\right)$$

and the general solution becomes:

$$\boxed{y(t) = C_1 e^{-2t} + C_2 e^{-4t} + \frac{3}{2} t e^{-2t} + \frac{1}{16}(4t - 3)}$$

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(8) Solve the Diff. Eq'n.

$$\ddot{y} + 3\dot{y} + 2y = \frac{1}{1+e^{t}}$$

Since the forcing term is not polynomial, sin(x), cos(x), or exponential, we have to use the Variation of parameter method:

Solving the homogeneous eq'n:

$$\ddot{y}_h + 3\dot{y}_h + 2y_h = 0$$

we look for solutions $y_h(t) = e^{st}$, since it is linear, homogeneous and const. coeff's Diff. Eq'n. Then:

The characteristic equation is.

$$r^2+3r+2=0$$

i.e. $(r+2)(r+1)=0 \quad \Rightarrow r_1=-2, \ r_2=-1.$

Then, $y_u(t)=c_1y_1(t)+c_2y_2(t)$

with $y_1(t)=e^{-2t}$, $y_2(t)=e^{-t}$.

Hence: $\dot{y}_1(t)=-2e^{-2t}$, $\dot{y}_2(t)=-e^{-t}$

Then, the Wronskian is:

$$W(t)=\det\begin{pmatrix} y_1(t) & y_2(t) \\ \dot{y}_1(t) & \dot{y}_2(t)\end{pmatrix}=\det\begin{pmatrix} e^{-2t} & e^{-t} \\ -2e^{-2t} & -e^{-t}\end{pmatrix}$$

i.e. $W(t)=e^{-3t}$.

Comparing $a(t)\ddot{y}+b(t)\dot{y}+c(t)y=f(t)$ w

with

$$\ddot{y}+3\dot{y}+2y=\frac{1}{1+e^t}.$$

$a(t)=1$, and $f(t)=\frac{1}{1+e^t}$

Then, the particular solution is:

$$y_p(t)=A(t)y_1(t)+B(t)y_2(t)$$

with

$$A(t)=-\int\frac{y_2(t)f(t)}{a(t)W(t)}dt \ ; \ B(t)=\int\frac{y_1(t)f(t)}{a(t)W(t)}dt.$$

Thus:

$$A(t) = -\int \frac{e^{-t}\left(\frac{1}{1+e^t}\right)}{1 \cdot e^{-3t}}\,dt = -\int \frac{e^{2t}}{1+e^t}\,dt.$$

$$= -\int \frac{y}{1+y}\,dy = -\int \frac{1+y-1}{1+y}\,dy = -\int 1 - \frac{1}{1+y}\,dy$$

$y = e^t.$

$$= -y + \log|1+y| + C_1 = -e^t + \log|1+e^t| + C_1.$$

$$B(t) = \int \frac{e^{-2t}\left(\frac{1}{1+e^t}\right)}{1 \cdot e^{-3t}}\,dt = \int \frac{e^t}{1+e^t}\,dt.$$

$$= \int \frac{y}{1+y}\frac{1}{y}\,dy = \int \frac{1}{1+y}\,dy = \log|1+y| + C_2$$

$$= \log|1+e^t| + C_2.$$

$y = e^t$

Thus:

$$y_p(t) = \left(\log|1+e^t| - e^t\right)e^{-2t} + \left(\log|1+e^t|\right)e^{-t}$$

$$y_p(t) = \left(\log|1+e^t|\right)\left(e^{-2t} + e^{-t}\right) - e^{-t}$$

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La solución general es entonces:

$$\boxed{y(t) = C_1 e^{-2t} + C_2 e^{-t} + \left(e^{-2t} + e^{-t}\right)\log|1+e^{t}| - e^{-t}}$$

TERCERA PARTE.

(10) Computing Hooke's constant:

$$k = \frac{\Delta F}{\Delta L} = \frac{10\,lb}{(7-5)\,ft} = \frac{10}{2}\,lb/ft = 5\,lb/ft.$$

New particle:

Now, $w = 8\,lb$ Then, $m = \frac{w}{g} = \frac{8\,lb}{32\,ft/sec^2} = \frac{1}{4}$ slug.

Finally $b = 1\,\frac{lb}{ft/sec}$.

$\uparrow y$

$y = 0$

Initial conditions: $y(0) = -1/2\,ft.$

$\dot{y}(0) = -1\,ft/sec.$

(a) $m\ddot{y} + b\dot{y} + ky = 0.$

$$\frac{1}{4}\ddot{y} + \dot{y} + 5y = 0 \Rightarrow \ddot{y} + 4\dot{y} + 20y = 0$$

$$y = e^{rt} \Rightarrow r^2 + 4r + 20 = 0 \Rightarrow r = \frac{-4 \pm \sqrt{16-80}}{2}$$

$$r = \frac{-4 \pm 4\sqrt{1-5}}{2} = \frac{-4 \pm 4(2i)}{2} = -2 \pm 4i$$

$$\Rightarrow \boxed{y(t) = \left(C_1\cos(4t) + C_2\sin(4t)\right)e^{-2t}}$$

Now: $\dot{y}(t) = 4(-C_1\sin(4t) + C\cos(4t))e^{-2t} + (-2)(C_1\cos(4t) + C_1\sin(4t))e$

$= (4)^2 =$

$$y(0) = C_1 = -\tfrac{1}{2} \text{ ft}$$

$$\dot{y}(0) = 4C_2 + (-2)C_1 = -1.$$

$$4C_2 + 1 = -1 \quad \Rightarrow \quad C_2 = \frac{-2}{4} = -\frac{1}{2} \text{ ft/sec.}$$

$$\boxed{y(t) = -\frac{1}{2}\Big(\cos(4t) + \sin(4t)\Big)e^{-2t}}$$

~~(b)~~ $A^2 = \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 = 2\,\frac{1}{4} \Rightarrow \boxed{A = \frac{\sqrt{2}}{2} \text{ ft}}$

$$A\cos\varphi = -\frac{1}{2} \quad \Rightarrow \quad \tan\varphi = -1 \quad \Rightarrow \quad \varphi = \frac{3\pi}{4}$$

$$A\sin\varphi = -\left(-\frac{1}{2}\right) \qquad\qquad \text{or } \varphi = -\frac{\pi}{4}$$

But $\cos\varphi < 0$, $\sin\varphi > 0 \Rightarrow \varphi = \frac{3\pi}{4}$

$$\boxed{y(t) = \frac{\sqrt{2}}{2}\cos\left(4t - \frac{3\pi}{4}\right)}$$

No se pide.

(b)

$$\boxed{y(10) = \frac{\sqrt{2}}{2}\cos\left(40 - \frac{3\pi}{4}\right)}$$

or $y(10) = -\frac{1}{2}\Big(\cos(40) + \sin(40)\Big)e^{-20}$

$$\approx 8.05 \times 10^{-11} \text{ ft}$$

(9). $m = 2\,kg$.

$$k = \frac{\Delta F}{\Delta L} = \frac{(2kg)g}{1/10\ m} = \frac{(20)^{kg}(9.8)\ m/s^2}{m} = \cancel{186}\ N. = 196\ N.$$

$$y(0) = \frac{5}{100}\ m$$
$$\dot{y}(0) = 1\ m/sec.$$

(a) $m\ddot{y} + b\dot{y} + ky = 0$ $\quad m = 2\,kg$
$b = 0\,(N/sec)\cdot m.$
$k = 196\,N.$

$$2\ddot{y} + 196y = 0.$$
$$\ddot{y} + 98y = 0$$
$$\boxed{y(t) = C_1\cos(\sqrt{98}\,t) + C_2\sin(\sqrt{98}\,t)}$$

Now $\dot{y}(t) = \sqrt{98}\left(-C_1\sin(\sqrt{98}\,t) + C_2\cos(\sqrt{98}\,t)\right)$

$$y(0) = C_1 = \frac{5}{100}\ m = \frac{1}{20}\ m$$
$$\dot{y}(0) = \sqrt{98}\,C_2 = 1\ m/sec$$
$\Rightarrow$ $\boxed{C_1 = 1/20\ m \quad C_2 = \frac{1}{\sqrt{98}}\ m/sec}$

$$\boxed{y(t) = \frac{1}{20}\cos(\sqrt{98}\,t) + \frac{1}{\sqrt{98}}\sin(\sqrt{98}\,t)}$$

$$y(t) = A\cos(\sqrt{98}\,t - \varphi)$$
$$y(t) = A\cos\varphi\cos(\sqrt{98}\,t) - A\sin\varphi\sin(\sqrt{98}\,t)$$

$\Rightarrow$ $A\cos\varphi = \frac{1}{20}$
$A\sin\varphi = -1/\sqrt{98}$

$\tan\varphi = \frac{-1/\sqrt{98}}{1/20} = \frac{-20}{\sqrt{98}}$

Since $\sin\varphi < 0$.
$\varphi \in [-\frac{\pi}{2}, 0]$

$$A^2 = \frac{1}{20^2} + \frac{1}{98} = \frac{98+400}{(400)98} = \frac{498}{39200}$$

$$\boxed{A = 0.11271 \text{ meters}}$$

$$\tan\varphi = \cancel{\quad} = \frac{-20}{\sqrt{98}} =$$

$$\varphi = -1.111 \text{ rad.}$$

Then.

$$\boxed{y(t) = 0.11 \cos\left(\sqrt{98}\,t + 1.11\right)}$$

or:
$$y(t) = 0.11 \cos\left(\sqrt{98}\,t + 1.11 + \frac{\pi}{2} - \frac{\pi}{2}\right)$$

$$y(t) = 0.11 \sin\left(\sqrt{98}\,t + 1.11 + \frac{\pi}{2}\right).$$

$$\boxed{y(t) \approx 0.11 \sin\left(\sqrt{98}\,t + 2.68\right)}$$

(c)

$$\omega = \sqrt{98}\ 1/\text{sec}$$

$$\boxed{A = 0.11 \text{ ft}}$$

$$\boxed{P = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{98}} \approx 0.63 \text{ sec.}}$$

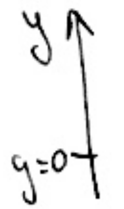
$$\cancel{f = \frac{1}{\omega} \approx 0.10105/\text{sec.}} \quad \boxed{f = \frac{1}{P} = 1.57 (1/\text{sec})}$$

In 5π sec: $\boxed{f \cdot 5\pi = 24.74 \text{ osallstrus}}$

(B1). $m = 1\,kg.$, $\Delta L = 0.098\,m.$

Hooke's constant: $k = \frac{\Delta F}{\Delta L} = \frac{m \cdot g}{9.8 \times 10^{-2}\,m} = \frac{(1kg)\,9.8\,m/sec^2}{9.8 \times 10^{-2}\,m}$

$= 100\ kg/sec^2 = 100\ N/m$



No damping $\therefore b = 0$

$y(0) = -0.1\,m$ External force:

$\dot{y}(0) = 0\ m/sec.$ $f(t) = 2\sin(10t)$.

(a) $m\ddot{y} + b\dot{y} + ky = f(t).$

$$\ddot{y} + 100y = 2\sin(10t)$$

$$y(0) = -\frac{1}{10}\,m$$

$$\dot{y}(0) = 0\ m/sec.$$

Homogeneous eq'n:

$$\ddot{y}_h + 100\,y_h = 0$$

with solution: $y_h(t) = C_1\cos(10t) + C_2\sin(10t)$

Particular solution:

$$y_p(t) = (\alpha\cos(10t) + \beta\sin(10t))\,t.$$

$$\dot{y}_p(t) = -10(\alpha\sin(10t) - \beta\sin(10t))\,t + (\alpha\cos(10t) + \beta\sin(10t))$$

$$\ddot{y}_p(t) = -10^2(\alpha\cos(10t) + \beta\sin(10t))\,t - 20(\alpha\sin(10t) - \beta\sin(10t))$$

Substitute into the equation:

$$\ddot{y} + 100y = 2\sin(10t)$$

$$-10^2\cancel{\left(\alpha\cos(10t) + \beta\sin(10t)\right)t} - 20\left(\alpha\sin(10t) - \beta\cos(10t)\right)$$

$$+ 100\cancel{\left(\alpha\cos(10t) + \beta\sin(10t)\right)} = 2\sin(10t)$$

$$\left.\begin{aligned} -20\alpha &= 2 \\ 20\beta &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} \alpha &= -\frac{1}{10} \\ \beta &= 0 \end{aligned}$$

The general solution is

$$y(t) = C_1\cos(10t) + C_2\sin(10t) - \frac{t}{10}\cos(10t)$$

$$\dot{y}(t) = 10\left(-C_1\sin(10t) + C_2\cos(10t)\right) - \frac{1}{10}\cos(10t) + \frac{t}{\cancel{10}}\sin(10t)$$

$$\begin{aligned} y(0) &= C_1 = -\frac{1}{10} \\ \dot{y}(0) &= 10C_2 - \frac{1}{10} = 0 \end{aligned} \Rightarrow \boxed{\begin{aligned} C_1 &= -1/10 \\ C_2 &= +1/100 \end{aligned}}$$

$$\boxed{y(t) = -\frac{1}{10}\cos(10t) + \frac{1}{100}\sin(10t) - \frac{t}{10}\cos(10t)}$$

(b) At $t = 20$:

$$\boxed{y(20) = -\frac{1}{10}\cos(200) + \frac{1}{100}\sin(200) - 2\cos(200)}$$

The term $-\frac{t}{10}\cos(10t)$ is the resonant term