Tarea #1 Solución | Viernes, Mayo 18, 2018

Se considera la ecuación de onda en coordenadas de laboratorio:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0.$$

con $u = u(x,t)$. Cambiando a coordenadas características. (también llamadas cónicas, nulas)

$$\xi(x,t) = x - ct$$
$$\eta(x,t) = x + ct,$$

encuentre la ecuación de onda en estas coordenadas.

Tenemos que calcular $\frac{\partial^2 u}{\partial t^2}$, $\frac{\partial^2 u}{\partial x^2}$ y sustituir en la ec. de onda.

Tenemos; por la regla de la cadena:

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t} = -c\frac{\partial u}{\partial \xi} + c\frac{\partial u}{\partial \eta}$$

Now:

$$\frac{\partial^2 u}{\partial t^2} = -c\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial \xi}\right) + c\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial \eta}\right)$$

$$= -c\left(\frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial t} + \frac{\partial^2 u}{\partial \eta \partial \xi}\frac{\partial \eta}{\partial t}\right) + c\left(\frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial t} + \frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial t}\right)$$

$$= -c\left(-c\frac{\partial^2 u}{\partial \xi^2} + c\frac{\partial^2 u}{\partial \eta \partial \xi}\right) + c\left(-c\frac{\partial^2 u}{\partial \xi \partial \eta} + c\frac{\partial^2 u}{\partial \eta^2}\right)$$

$$= c^2\frac{\partial^2 u}{\partial \xi^2} - c^2\frac{\partial^2 u}{\partial \eta \partial \xi} - c^2\frac{\partial^2 u}{\partial \xi \partial \eta} + c^2\frac{\partial^2 u}{\partial \eta^2}$$

$$= c^2 u_{\xi\xi} - 2c^2 u_{\eta\xi} + c^2 u_{\eta\eta}$$

Similarly,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \xi}\right) + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \eta}\right) = \frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial x} + \frac{\partial^2 u}{\partial \eta \partial \xi}\frac{\partial \eta}{\partial x}$$

$$+ \frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial x} + \frac{\partial^2}{\partial \eta^2} u \frac{\partial \eta}{\partial x}.$$

$$= \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta \partial \xi} + \frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}$$

$$= \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi . \eta} + \frac{\partial^2 u}{\partial \eta^2}$$

Hence:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = \left(\cancel{c^2 \frac{\partial^2}{\partial \xi^2} u} - 2c^2 \frac{\partial^2}{\partial \xi \partial \eta} u + \cancel{c^2 \frac{\partial^2}{\partial \eta^2} u}\right)$$

$$- c^2\left(\cancel{\frac{\partial^2 u}{\partial \xi^2}} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \cancel{\frac{\partial^2 u}{\partial \eta^2}}\right)$$

$$\underbrace{\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2}}_{=0} = -4c^2 \frac{\partial^2 u}{\partial \xi \partial \eta}$$

. Thus: $\boxed{\frac{\partial^2 u}{\partial \xi \partial \eta} = 0}$

Tarea #1 Solución BIS | Viernes, Mayo 18, 2018.

Se considera la ec. de onda:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 .$$

y donde $u = u(x,t)$. Se considera el cambio de variables.

$$\xi(x,t) = x - ct$$
$$\eta(x,t) = x + ct.$$

Demostrar que: $\dfrac{\partial^2 U}{\partial \eta \partial \xi} = 0.$

Definimos.

$$U = U(\xi, \eta) = u(x(\xi,\eta), y(\xi,\eta)).$$

We have to compute: $\frac{\partial U}{\partial \xi}$ using the chain rule:

$$\frac{\partial U}{\partial \xi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial \xi} = \frac{1}{2}\frac{\partial u}{\partial x} + \left(-\frac{1}{2c}\right)\frac{\partial u}{\partial t} = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{1}{c}\frac{\partial u}{\partial t}\right)$$

puesto

$$x(\xi,\eta) = \frac{1}{2}(\xi+\eta)$$
$$t(\xi,\eta) = \frac{1}{2c}(\eta-\xi)$$

Now, we have to compute $\frac{\partial^2 U}{\partial \eta \partial \xi} = \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi}\right) =$

$$= \frac{\partial}{\partial \eta}\left(\frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{1}{c}\frac{\partial u}{\partial t}\right)\right) = \frac{1}{2}\frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial x}\right) - \frac{1}{2c}\frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial t}\right)$$

$$= \frac{1}{2}\left(\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\frac{\partial x}{\partial \eta} + \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)\frac{\partial t}{\partial \eta}\right) - \frac{1}{2c}\left(\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right)\frac{\partial x}{\partial \eta} + \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right)\frac{\partial t}{\partial \eta}\right)$$

$$= \frac{1}{2}\left(\frac{\partial^2 u}{\partial x^2}\cdot\frac{1}{2} + \frac{\partial^2 u}{\partial t \partial x}\left(\frac{1}{2c}\right)\right) - \frac{1}{2c}\left(\frac{\partial^2 u}{\partial x \partial t}\frac{1}{2} + \frac{\partial^2}{\partial t^2}u\left(\frac{1}{2c}\right)\right)$$

$=\!\!\int =$

$$= \frac{1}{4}\frac{\partial^2 u}{\partial x^2} + \frac{1}{4c}\frac{\partial^2 u}{\partial t \partial x} - \frac{1}{4c}\frac{\partial^2 u}{\partial x \partial t} - \frac{1}{4c^2}\frac{\partial^2}{\partial t^2} u$$

$$= \frac{1}{4c^2}\Big(\underbrace{c^2 \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2}{\partial t^2} u}_{=0}\Big) + \frac{1}{4c}\Big(\underbrace{\frac{\partial^2 u}{\partial t \partial x} - \frac{\partial^2 u}{\partial x \partial t}}_{=0}\Big)$$

pues u satisface la Ec. de onda

pues $\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right)$ si $u \in C^2$.

$= 0$ Entonces $\boxed{\frac{\partial^2 U}{\partial \eta \partial \xi} = 0}$
