Problema Q de la Tarea #2.

Definimos la Transformada de Fourier de la función (integrable) f como:

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx.$$

y la Transformada inversa de Fourier de f F(k) como:

$$\check{F}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k)\, e^{ikx}\, dk$$

Suponer que $f(x) \xrightarrow[x \to \pm\infty]{} 0$.

Demostrar las siguientes propiedades de la T. de Fourier:

(a) $\frac{d}{dk}\left(\hat{f}(k)\right) = -i\, \widehat{x f(x)}(k)$

(b) $\widehat{\frac{df}{dx}}(k) = ik\, \hat{f}(k)$.

(c) $\int \hat{f}(k)\, dk = i\, \widehat{\frac{f(x)}{x}}(k)$

(d) $\widehat{\int f(x)\, dx}(k) = \frac{1}{ik}\, \hat{f}(k)$

(e) $\widehat{f(x+x_0)}(k) = \hat{f}(k)\, e^{x_0 i k}$.

Temas Selectos de Ingeniería Física [UAM-Azcapotzalco] 18-P

Intro a las Ondas lineales, no lineales y solitones.

Tarea # 2. Fecha de entrega: 25 de mayo de 2018

(1) Considere una Ecuación diferencial parcial (PDE), lineal, de coeficientes constantes.

$$i\frac{\partial\varphi}{\partial t} = W\left(-i\frac{\partial}{\partial x}\right)\varphi$$

Usando la Transformada de Fourier, y suponiendo que $W(z)$ es un polinomio real, de variable z, deduzca que la solución de la PDE está dada por:

$$\varphi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{F}(k)\, e^{i(kx - W(k)t)}\, dk,$$

en donde $\hat{F}(k)$ es la T. de Fourier de la condición inicial:

$$\varphi(x,0) = f(x).$$

Nota: La T. de Furier y su inversa son:

$$\hat{f}(k) = F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$$

and $$\check{F}(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)\, e^{ikx}\, dk$$

Recordar que:
$$\widehat{-i\frac{d}{dx}f}(k) = k\hat{f}(k)$$

Consider the $\boxed{\text{Homework}}$ case where $W_2(k) = -W_1(k) = -W(k)$

and $W_1(k) = W(k)$

Then:

$$\sqrt{2\pi}\, \varphi(x,t) = \int_{-\infty}^{\infty} F_1(k) e^{i(kx - W(k)t)}\, dk + \int_{-\infty}^{\infty} F_2(k) e^{i(kx + W(k)t)}\, dk$$

With the initial conditions:

$$\varphi(x,0) = f(x).$$

$$\partial_t \varphi(x,0) = g(x).$$

(a) Use the initial conditions to get:

$$\sqrt{2\pi}\, f(x) = \int_{-\infty}^{\infty} \left(F_1(k) + F_2(k)\right) e^{ikx}\, dk = \sqrt{2\pi}\left(F_1(k) + F_2(k)\right)^{\vee}$$

$$\sqrt{2\pi}\, g(x) = -i \int_{-\infty}^{\infty} W(k)\left(F_1(k) - F_2(k)\right) e^{ikx}\, dx = \sqrt{2\pi}\left(W(k)(F_1 - F_2)\right)^{\vee}$$

(b) Apply the Fourier Transform to get:

$$\left.\begin{aligned} \hat{f}(k) &= F_1(k) + F_2(k) \\ \hat{g}(k) &= -i W(k)\left(F_1(k) - F_2(k)\right) \end{aligned}\right\} (*)$$

(Notice that the Right-Hand-Side of $(*)$ is already the Inverse Fourier transform, so we do not

anything to do:

(c) Solve $(*)$ for F_1 and F_2 to get:

$$F_1(k) = \frac{1}{2}\left[\hat{f}(k) + i\frac{\hat{g}(k)}{W(k)}\right]$$

$$F_2(k) = \frac{1}{2}\left[\hat{f}(k) - i\frac{\hat{g}(k)}{W(k)}\right]$$

(d) Since $f(x)$ and $g(x)$ are real,

Show that
$$\left(\hat{f}(k)\right)^* = \hat{f}(-k)$$
$$\left(\hat{g}(k)\right)^* = \hat{g}(-k).$$

$(\ )^*$ = complex conjugate of $(\ )$.

(e). Show that, if $W(k)$ is odd, $W(-k) =$, then:

$$F_1^*(k) = F_1(-k).$$

$$F_2^*(k) = F_2(-k).$$

(f) Using again that $W(k)$ is odd and item (e)
Compute φ^* and show that:

$$\varphi^* = \varphi.$$

i.e., the solution φ is real:

(h) Now, if $W(k)$ is even, $W(-k) = W(k)$, show that:

$$F_1^*(k) = F_2(-k)$$
$$F_2^*(k) = F_1(-k)$$

(g) Again, $W(k)$ is even. Using (h), ~~show~~ Compute φ^*, and show: $\varphi^* = \varphi$

Then, $\varphi(x,t)$ is real, the sol'n is real