TAREA #2 (Soluciones). Viernes 25 de agosto de 2018.

Temas Selectos de Ingeniería Física II.

LINEAR, NONLINEAR WAVES AND SOLITON

Problema 01 This is a review of the Fourier transform,

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx.$$

The Inverse Fourier Transform is:

$$\check{F}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k)\, e^{ikx}\, dk$$

Fourier proved that (Fourier theorem):

$$\check{(\hat{f})}(x) = f(x)$$

Assume $f(x) \xrightarrow[x \to \pm\infty]{} 0$

(a) Here, we assume that $\frac{\partial}{\partial k}\left(f(x)\, e^{-ikx}\right) = -ix f(x) e^{-ikx}$ is integrable, to apply Leibnitz rule:

$$\text{Then: } \frac{d}{dk}\left(\hat{f}(k)\right) = \frac{1}{\sqrt{2\pi}} \frac{d}{dk} \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx \overset{\text{Leibnitz}}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial}{\partial k}\left(f(x) e^{-ikx}\right) dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -ix f(x) e^{-ikx}\, dx = (-i) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (x f(x))\, e^{-ikx}\, dx$$

$$= (-i)\, \widehat{x f(x)}\,(k) \quad \checkmark \quad \text{Q.E.D.}$$

(b) Here, we assume $\frac{df}{dx}$ is integrable, so that its Fourier T. exists:

$$\widehat{\frac{df}{dx}}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{df}{dx}(x)\, e^{-ikx}\, dx = \frac{1}{\sqrt{2\pi}} \Bigg[\underbrace{f(x) e^{-ikx} \Big|_{x=-\infty}^{x=\infty}}_{=0} - \int_{-\infty}^{\infty} f(x) \frac{d}{dx} e^{-ikx} dx$$

$$= \frac{1}{\sqrt{2\pi}} (-1) \int_{-\infty}^{\infty} f(x)(-ik) e^{-ikx} dx = (ik) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

$$= ik \hat{f}(k)$$

(c) Here we assume: $\frac{f(x)}{x}$ is integrable, so that its Fourier T. exists and we can use Leibnitz rule.

Let $G(k) \equiv \int \hat{f}(k) dk$ be the antiderivative of $\hat{f}(k)$.

Then: $\frac{dG}{dk}(k) = \hat{f}(k)$. Assume $G(k)$ is the Fourier T. of some function $g(x)$: $G(k) = \hat{g}(k)$. Then:

$$\frac{dG}{dk} = \frac{d}{dk} \hat{g}(k) = (-i)\, \widehat{xg(x)}(k), \text{ by prop. (a).}$$

Thus:

$$\hat{f}(k) = \frac{dG}{dk} = -i\, \widehat{(xg(x))}(k).$$

Then by Fourier theorem:

$$f(x) = -ixg(x). \Rightarrow g(x) = i\frac{f(x)}{x}$$

Hence:

$$\int \hat{f}(k) dk = G(k) = \hat{g}(k) = i \widehat{\left(\frac{f(x)}{x}\right)}(k),$$

(d) Let $H(x) = \int f(x)\,dx$. Hence: $\frac{dH}{dx} = f(x)$.

By property (b):

$$\widehat{\frac{dH}{dx}}(k) = ik\,\widehat{H}(k). \Rightarrow \widehat{H}(k) = \frac{1}{ik}\widehat{\frac{dH}{dx}}(k)$$

Hence:

$$\widehat{\int f(x)dx} = \widehat{H}(k) = \frac{1}{ik}\widehat{\frac{dH}{dx}} = \frac{1}{ik}\widehat{f}(k) \checkmark$$

(e)
$$\widehat{f(x+x_0)}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x+x_0)\,e^{-ikx}\,dx \overset{y = x+x_0}{=} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(y)\,e^{-ik(y-x_0)}\,dy$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(y)\,e^{-iky}e^{ikx_0}\,dy = \frac{1}{\sqrt{2\pi}}\int_{-\infty} f(y)e^{-iky}dy\,e^{ik}$$

$$= \widehat{f}(k)\,e^{ikx_0}. \checkmark$$

Problem (4). We want to solve the linear differential equation with constant coefficients:

$$i\frac{\partial\varphi}{\partial t} = W\left(-i\frac{\partial}{\partial x}\right)\varphi$$

with initial condition:

$$\varphi(x,0) = f(x).$$

We assume $W(z)$ is a polynomial, real, of constant coefficients, of odd power (or with imaginary coeff. in terms of even power)

To fix ideas, set

$$W(z) = \alpha z + \beta z^3$$

Hence:

$$W\left(-i\frac{\partial}{\partial x}\right)\varphi = \left[\alpha\left(-i\frac{\partial}{\partial x}\right) + \beta\left(-i\frac{\partial}{\partial x}\right)^3\right]\varphi$$

$$= \left[-i\alpha\frac{\partial}{\partial x} + i\beta\frac{\partial^3}{\partial x^3}\right]\varphi$$

$$= -i\alpha\frac{\partial\varphi}{\partial x} + i\beta\frac{\partial^3\varphi}{\partial x^3}.$$

Hence, the Diff. Eqn becomes:

$$i\frac{\partial\varphi}{\partial t} = -i\alpha\frac{\partial\varphi}{\partial x} + i\beta\frac{\partial^3}{\partial x^3}\varphi \quad \cdots\cdots (*)$$

Now: $$\widehat{\frac{\partial\varphi}{\partial t}}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial\varphi}{\partial t}(x,t)\,e^{-ikx}\,dx = \frac{d}{dt}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\varphi(x,t)e^{-ikx}dx\right]$$

$$= \frac{d}{dt}\hat{\varphi}(k,t)$$

Then applying the Fourier T. to each member of $(*)$

$$\widehat{i\frac{\partial\varphi}{\partial t}} = -i\alpha\widehat{\frac{\partial\varphi}{\partial x}} + i\beta\widehat{\frac{\partial^3\varphi}{\partial x^3}}$$

$$= -i\alpha\, ik\hat{\varphi}(k) + i\beta(ik)^3\hat{\varphi},$$ by Prop (b)

i.e.

$$i\frac{d\hat{\varphi}}{dt} = \alpha k\hat{\varphi}(k) + \beta k^3\hat{\varphi}(k)$$

$$= (\alpha k + \beta k^3)\hat{\varphi}(k)$$

$$= W(k)\hat{\varphi}(k).$$

For a general polynomial $W(k)$, we can conclude,

$$i\frac{\partial\varphi}{\partial t} = W\left(-i\frac{\partial}{\partial x}\right)\varphi,$$

is transformed into

$$\widehat{i\frac{\partial\varphi}{\partial t}} = \widehat{W\left(-i\frac{\partial}{\partial x}\right)\varphi}$$

i.e. $$\boxed{i\frac{d}{dt}\hat{\varphi} = W(k)\hat{\varphi}.}$$

and we will work with this expression instead of working with $W(k) = \alpha k + \beta k^3$.

The initial conditions transform into $\hat{\varphi}(k,0) = \hat{f}(k)$.

Then, we have the Initial Value Problem.

$$\frac{d}{dt}\hat{\varphi} = -iW(k)\hat{\varphi}$$

$$\hat{\varphi}(k,0) = \hat{f}(k)$$

with solution.

$$\hat{\varphi}(k,t) = \hat{f}(k)\, e^{-iW(k)t}$$

By Fourier Thm.

$$\varphi(x,t) = \check{(\hat{\varphi})} = \check{\left(\hat{f}(k)\, e^{-iW(k)t}\right)}$$

i.e.

$$\varphi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\hat{f}(k)\, e^{-iW(k)t}\right) e^{ikx}\, dk.$$

i.e. $$\boxed{\varphi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(k)\, e^{i(kx - W(k)t)}\, dk}$$

Problem ②: If the dispersion relation $R(k,\omega)=0$, has two solutions,

$$\omega = W_1(k) = W(k)$$
$$\omega = W_2(k) = -W_1(k) = -W(k)$$

Since we assume our Diff. Eq'n is linear, then,

$$\varphi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_1(k)\, e^{i(kx - W_1(k)t)}\, dk + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_2(k)\, e^{i(kx - W_2(k)t)}\, dk$$

The initial conditions are:

$$\varphi(x,0) = f(x)$$
$$\partial_t \varphi(x,0) = g(x)$$

(a) Computing:

$$\partial_t \varphi = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -iW_1(k) F_1(k)\, e^{i(kx - W_1(k)t)}\, dk + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -iW_2(k) F_2(k)\, e^{i(kx - W_2(k)t)}\, dk.$$

and evaluating at $t=0$.

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_1(k)\, e^{ikx}\, dk + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_2(k)\, e^{ikx}\, dk$$

$$g(x) = \left(-\frac{i}{\sqrt{2\pi}}\right) \left[\int_{-\infty}^{\infty} W_1(k) F_1(k) e^{ikx}\, dk + \int_{-\infty}^{\infty} W_2(k) F_2(k) e^{ikx}\, dk \right]$$

Since $W_1(k) = W(k)$, $W_2(k) = -W(k)$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(F_1(k) + F_2(k)\right) e^{ikx}\, dk$$

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -iW(k)\left(F_1(k) - F_2(k)\right) e^{ikx}\, dk.$$

(b) The Right-Hand-Side of both eq'n is the Inverse Fourier transform. Then, by the Fourier Theorem:

$$\hat{f}(k) = F_1(k) + F_2(k)$$

$$\hat{g}(k) = -iW(k)\left(F_1(k) - F_2(k)\right)$$

(c) Now:

$$F_1(k) + F_2(k) = \hat{f}(k)$$

$$F_1(k) - F_2(k) = \frac{i}{W(k)} \hat{g}(k)$$

Then:

$$F_1(k) = \frac{1}{2}\hat{f}(k) + \frac{i}{2W(k)} \hat{g}(k).$$

and

$$F_2(k) = \frac{1}{2}\hat{f}(k) - \frac{i}{2W(k)} \hat{g}(k).$$

(d) We have:

$$\left(\hat{f}(k)\right)^* = \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx\right)^* =$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f^*(x)\, e^{ikx}\, dk \underset{f^* = f}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i(-k)x}\, dk$$

$k = -(-k)$

$= \hat{f}(-k)$, the Fourier T. evaluated at $-k$.

Hence: $(\hat{f}(k))^* = \hat{f}(-k)$

and similarly $(\hat{g}(k))^* = \hat{f}(-k)$.

(e) If $W(k)$ is odd: $W(-k) = -W(k)$, then:

$$F_1^*(k) = \left(\frac{1}{2}\hat{f}(k) + \frac{i}{2W(k)}\hat{g}(k)\right)^* = \frac{1}{2}(\hat{f}(k))^* - \frac{i}{2W^*(k)}(\hat{g})^*$$

$$\underset{\text{Assuming } W \text{ is real}}{=} \frac{1}{2}\hat{f}(-k) + \frac{i}{-2W(k)}\hat{g}(-k) \underset{W \text{ is odd}}{=} \frac{1}{2}\hat{f}(-k) + \frac{i}{2W(-k)}\hat{g}(-k)$$

$$= F_1(-k)$$

Similarly $F_2^*(k) = F_2(-k)$.

(f). We have to compute:

$-W_2(k) = W(k)$

$$\varphi^*(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_1^*(k)\, e^{-i(kx - W(k)t)}\, dk + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_2^*(k)\, e^{-i(kx + W(k)t)}\, dk$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_1(-k)\, e^{i[(-k)x - W(-k)t]}\, dk + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_2(-k)\, e^{i[(-k)x + W(-k)t]}\, dk$$

W is odd

$F_{1,2}^*(k) = F_{1,2}(-k)$

Changing variables $K = -k$.

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_1(K)\, e^{i[Kx - W(K)t]}\, dK + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_2(K)\, e^{i[Kx + W(K)t]}\, dK$$

$$= \varphi(x,t) \Rightarrow \varphi^* = \varphi \Rightarrow \underline{\varphi \text{ is real.}}$$

Remark: $\displaystyle\int_{-\infty}^{\infty} f(-k)\, dk \underset{K=-k}{=} \int_{+\infty}^{-\infty} f(K)(-1)\, dK = -\int_{+\infty}^{\infty} f(K)\, dK$

$$= \int_{-\infty}^{\infty} f(K)\, dK, \text{ was used above.}$$

(h) Now: $W(-k) = W(k)$. Then:

$$F_1^*(k) = \frac{1}{2}(\hat{f}(k))^* - \frac{i}{2W(k)}(\hat{g}(k))^* = \frac{1}{2}\hat{f}(-k) - \frac{i}{2W(-k)}\hat{g}(-k)$$

W is even.

$$= F_2(-k).$$

Similarly $F_2^*(k) = F_1(-k)$.

(g) $$\varphi^*(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_1^*(k)\, e^{-i[kx - W(k)t]}\, dk + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_2^*(k)\, e^{-i[kx + W(k)t]}\, dk$$

$=?=$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_2(-k) e^{i((-k)x + W(-k)t)} dk + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_1(-k) e^{i((-k)x - W(-k)t)} dk$$

By (4), and W is even.

Now, taking $k = -k$, and using the previous remark:

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_2(k) e^{i(kx + W(k)t)} dk + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_1(k) e^{i(kx - W(k)t)} dk.$$

$= \varphi(x,t)$. Then $\varphi(x,t)$ is a real solution