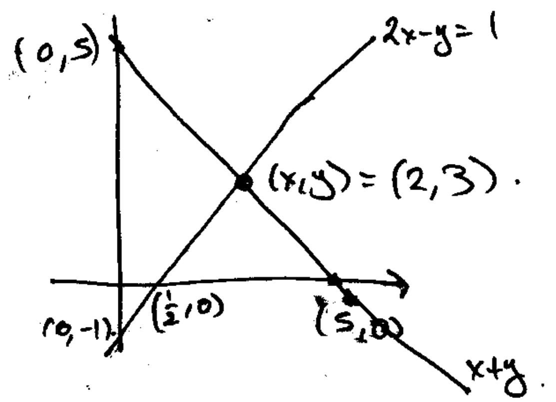
1. ~~Introduction~~ Review to Linear Algebra

Consider the system of equations:

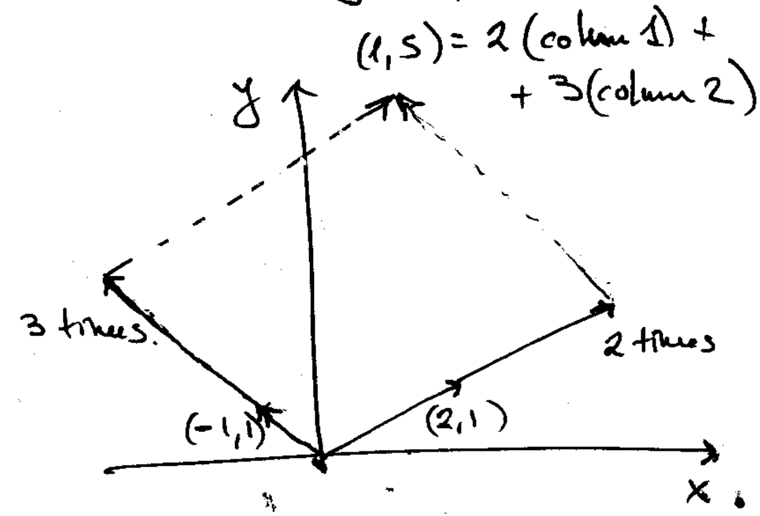
$$\begin{aligned} 2x - y &= 1 \\ x + y &= 5 \end{aligned}$$ with solution: $x = 2$, $y = 3$

The solution represents the intersection of the two lines:



Similarly, we can represent the same system, as follows:

$$x\begin{pmatrix}2\\1\end{pmatrix} + y\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}1\\5\end{pmatrix}$$



Find the combination of the vectors on the LHS, to get the vector on RHS.

LHS = Left-hand-side

RHS = Right-hand-side

Now, the system of 3 eqs:

$$\begin{aligned} 2u + v + w &= 5 \\ 4u - 6v \quad &= -2 \\ -2u + 7v + 2w &= 9 \end{aligned} \qquad \text{------} \; (*)$$

Represent three planes in the 3-Dimensional space, whose solution $u = 1$, $v = 1$, $w = 2$ represents the intersection between them.

Similarly, the same system

$$u \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} + v \begin{pmatrix} 1 \\ -6 \\ 7 \end{pmatrix} + w \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 9 \end{pmatrix}$$

represents the combination of the vectors in the LHS to obtain the vector in the RHS.

<u>Matrix notation</u>. This system can be represented in matrix form:

$$\begin{pmatrix} 2 & 1 & 1 \\ -4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 9 \end{pmatrix} \qquad (**)$$

Similarly:

$$\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

represents the system of two eqs (previous page).

A **matrix** is an array of numbers (real or complex).

(array: of $n \times m$)

A **column vector** is a $n \times 1$ matrix. (n is a positive integer)

$$\begin{pmatrix} 5 \\ -2 \\ 9 \end{pmatrix}$$

A **row vector** is a $1 \times n$ matrix

$$\begin{pmatrix} 3 & 2 & -5 \end{pmatrix}$$

Vector = Column vector. (I usually use this convention)

Row vector = Row vector.

Multiplication of matrices

Matrix notation:

Consider the system (*) in page =2=.

The unknown is $\bar{x} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$

The solution is $\bar{x} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$. Call $\bar{b} = \begin{pmatrix} 5 \\ -2 \\ 9 \end{pmatrix}$.

The nine coefficients can be written as

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix}$$

Then eq (**) becomes:

$$\boxed{A\bar{x} = \bar{b}}.$$

Matrix notation of system (*).

Multiplication of a row with a column vector:

$$(2\ 1\ 1)\begin{pmatrix}u\\v\\w\end{pmatrix} = (2u+v+w)$$

Then, the product of a matrix by a vector is:

$A\bar{x} =$

$$= \begin{pmatrix}2 & 1 & 1\\ 4 & -6 & 0\\ -2 & 7 & 2\end{pmatrix}\begin{pmatrix}u\\v\\w\end{pmatrix} = \begin{pmatrix}(2u+v+w)\\(4u-6v)\\(-2u+7v+2w)\end{pmatrix}.$$

Similarly:

$$A\bar{x} = u\begin{pmatrix}2\\4\\-2\end{pmatrix} + v\begin{pmatrix}1\\-6\\7\end{pmatrix} + w\begin{pmatrix}1\\0\\2\end{pmatrix}$$ is a __combination of the columns of A.__

The coefficients of the columns are the components of $x = \begin{pmatrix}u\\v\\w\end{pmatrix}$.

Notice that the multiplication was

$$(3\times 3 \text{ matrix}) \text{ times } (3\times 1 \text{ matrix}) = 3\times 1 \text{ matrix}$$

In general, we can multiply:

$$(m\times k \text{ matrix}) \text{ times } (k\times n \text{ matrix}) = m\times n \text{ matrix}.$$

Notice this is __not__ commutative, ie $Ax \neq xA$.

Subscript notation:

jth component of $\bar{x} = x_j$

ijth element of $\mathbf{A} = a_{ij}$ ← Located at the ith row and the jth column.

Then, the product $A\bar{x}$ can be written as.

$$\sum_{j=1}^{n} a_{ij} x_j \text{ is the } i^{th} \text{ component of } Ax$$

$$\sum_{j=1}^{n} a_{ij} x_j = a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n.$$

Multiplication of matrices - Revisited.

$$AB = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ b_{41} & b_{42} \end{pmatrix} = \begin{pmatrix} * & * \\ * & * \\ * & (AB)_{32} \end{pmatrix}$$

3 x 4 matrix 4 x 2 matrix = 3 x 2 matrix

$$a_{31} b_{12} + a_{32} b_{22} + a_{33} b_{32} + a_{34} b_{42} = (AB)_{32}$$

$$\sum_{i=1}^{4} a_{3i} b_{i2} = (AB)_{32}$$

Then, the ij-entry of AB is given by:

$$(AB)_{ij} = \sum_{k=1}^{m} A_{ik} B_{kj}$$

If B has columns $(b_1, b_2, \dots, b_n) = B$.
vectors.

Then,

$$AB = (Ab_1, Ab_2, \dots, Ab_n)$$

This is consistent with the matrix-vector multiplication.

If A has rows $A = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$

Then $AB = \begin{pmatrix} a_1 B \\ a_2 B \\ \vdots \\ a_n B \end{pmatrix}$

Matrix multiplication is associative

$$(AB)C = A(BC).$$

and distributive:

$$A(B+C) = AB + AC.$$

but rarely is commutative. I.e, is non-commutative,

i.e., usually $AB \neq BC$.

The matrix A is invertible, if there exists a matrix B such that $AB = I$ = Identity matrix $= \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{pmatrix}$

and $BA = I$

B is called the <u>inverse of A</u>, denoted by $B \equiv A^{-1}$.

Then:

$$AA^{-1} = I \quad \text{and} \quad A^{-1} = I.$$

(a) Then, the system $Ax = b$ has <u>a</u> solution if A is invertible. The solution is given by:

$$x = A^{-1} b.$$

(b) The inverse is unique.

Assume two inverses B and C of A.

$$B = B(AC) = (BA)C = C, \text{ the same.}$$

(c) What is the inverse of the matrix A^{-1}?

R: It is A.

(d) A 1×1 matrix is invertible if it is $\neq 0$.

$$A = (a), \quad A^{-1} = \left(\tfrac{1}{a}\right)$$

(e) A 2×2 matrix is invertible if $ad - cb \neq 0$, where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \text{ Then } A^{-1} = \frac{1}{ad-cb}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

(f) A diagonal matrix $\Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{pmatrix}$ is invertible,

if all of its entries are non-zero.

$$\Lambda^{-1} = \begin{pmatrix} \frac{1}{\lambda_1} & & & 0 \\ & \frac{1}{\lambda_2} & & \\ & & \ddots & \\ 0 & & & \frac{1}{\lambda_n} \end{pmatrix}$$

(g) The matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is not invertible.

This columns are "<u>linearly dependent</u>".

If A and B are invertible, and AB is defined, then, the inverse of AB is given by:

$$(AB)^{-1} = B^{-1} A^{-1}.$$

<u>The transpose of a matrix</u>:

If $A = \begin{pmatrix} 2 & 1 & 4 \\ 0 & 0 & 3 \end{pmatrix}$ then $A^t = \begin{pmatrix} 2 & 0 \\ 1 & 0 \\ 4 & 3 \end{pmatrix}$ is the transpose of A.

$$\boxed{(A^t)_{ij} = A_{ji}}$$

<u>Properties</u>: If AB is defined, $(AB)^t = B^t A^t$

If A^{-1} exists, its transpose is $(A^{-1})^t = (A^t)^{-1}$.

Vector Spaces and Subspaces.

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Def: A <u>real</u> (<u>complex</u>) <u>vector space</u> is a set of "vectors" together with rules for vector addition and multiplication by real numbers. (or complex numbers).

These rules satisfy the following properties

For all x, y, z in the vector space:

1) $x + y = y + x$
2) $x + (y + z) = (x + y) + z$
3) $\exists\ \bar{0}$ such that $x + \bar{0} = \bar{0} + x \quad \forall x$
4) $\forall x$, there is a $-x$, such that $x + (-x) = 0$
5) $1 \cdot x = x$
6) $(c_1 c_2) x = c_1 (c_2 x)$
7) $c(x + y) = cx + cy$
8) $(c_1 + c_2) x = c_1 x + c_2 x$.

c_1, c_2, c are scalars (real or complex)

Def. A <u>subspace</u> of a vector space is a nonempty subset that satisfies two requirements.

(i) The vectors x and y are in the subspace, implies $x + y$ is in the subspace

(ii) The vector x is in the subspace, then αx is in the subspace, for α scalar.

A <u>subspace</u> is a closed subset of the vector space under addition and scalar multiplication.

LINEAR DEPENDENCE, BASIS AND DIMENSION.

<u>Def.</u> Let $v_1, v_2, \dots, v_k$ vectors in $\mathbb{R}^n$. $(k \le n)$

If only the trivial combination of these vectors gives zero, this is to say, if

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0 \quad \text{only happens when}$$

$$c_1 = c_2 = \cdots = c_k = 0,$$

then the vectors are said to be <u>linearly independent</u>. Otherwise, they are linearly dependent, and one of them is the linear combination of the others.

<u>Example</u>. The columns of the matrix

$$A = \begin{pmatrix} 3 & 4 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{pmatrix}$$

are linearly independent.

Look for a combination of the columns.

We have to show that c_1, c_2, c_3 are <u>all</u> <u>forced</u> to be zero.

$$c_1 \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

i.e.

$$\begin{pmatrix} 3 & 4 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

i.e.

$$\boxed{\begin{aligned} 3c_1 + 4c_2 + 2c_3 &= 0 \\ c_2 + 5c_3 &= 0 \\ 2c_3 &= 0 \end{aligned}}$$

$3c_1 = 0 \Rightarrow c_1 = 0$.

$\Rightarrow c_2 = 0$

$\Rightarrow c_3 = 0$

Then, $c_1 = c_2 = c_3$, and the three columns are <u>linearly independent</u>.

A set of m vectors of $\mathbb{R}^n$ must be linearly dependent if $m > n$.

<u>Spanning</u>.

<u>Def</u>. If a vector space V consists of all the linear combinations of the particular vectors $w_1, w_2, \ldots, w_\ell$, then, we say these vectors <u>span</u> the vector space V. In other words, every vector v in V can be expressed as some combination of the w's:

$$v = \alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_\ell w_\ell, \quad \text{for some coefficients } \alpha_k \in \mathbb{R}$$

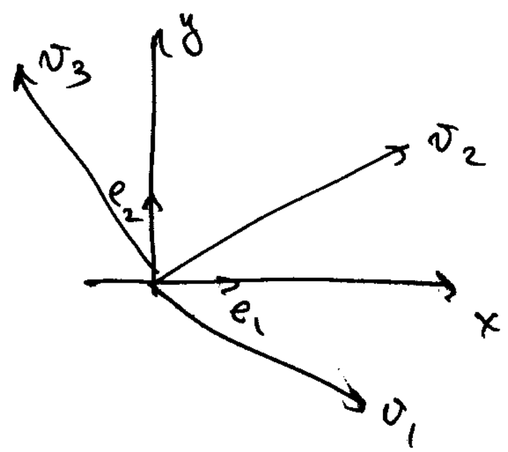
<u>Def</u> A <u>basis</u> for a vector space V is a set of vectors such that:

(a) It is linearly independent

(b) It spans the space.

Remark The basis of a vector space is not unique.
A vector space has infinitely many different basis.

Example $\mathbb{R}^2$



$\{v_1\}$ is by itself linearly independent, but does not ~~eq~~ span $\mathbb{R}^2$.

$\{v_1, v_2, v_3\}$ span $\mathbb{R}^2$, but they are not independent.

Any two of these vectors, say $\{v_2, v_3\}$
- span $\mathbb{R}^2$
- are linearly independent

Th Then, they form a basis for $\mathbb{R}^2$.

Similarly for $\{v_1, v_2\}$ $\{v_1, v_3\}$, $\{e_1, v_2\}$, $\{e_1, e_2\}$, ...
then, ~~the~~ a basis is not unique.

Dimension

Def. Any two bases of a vector space V contain the same number of vectors. This number (which is shared by all the bases and express the number of "degrees of freedom" of the space) is called the dimension of V.

I.e., if $v_1, \dots, v_m$ and $w_1, \dots, w_n$ are both bases for the same vector space V, then $m = n$.

THE FOUR FUNDAMENTAL SUBSPACES.

1. The column space of A: $R(A) = \{x \in V \mid x = Ac,\ c \in \mathbb{R}^n\}$
2. The null space of A: $N(A) = \{x \in V \mid Ax = 0\}$.
3. The row space of A: $R(A^t)$.
4. The left null space of A: $N(A^t)$

Def. The rank of a matrix A is the number of linearly independent columns of A, and it is denoted by r.

If A is a $m \times n$ matrix:

$N(A)$ and $R(A^T)$ are subspaces of $\mathbb{R}^n$.

$N(A^T)$ and $R(A)$ are subspaces of $\mathbb{R}^m$.

The rows have n components, and the columns m.

Example:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Column sp. $R(A) = \{\lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix} / \lambda \in \mathbb{R}\}$

left-Null sp. $N(A^T) = \{\lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix} / \lambda \in \mathbb{R}\}$.

Row sp. $R(A^T) = \{\lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} / \lambda \in \mathbb{R}\}$

Null sp. $N(A) = \{\lambda \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} / \lambda, \mu \in \mathbb{R}\}$

The Fundamental Theorem of Linear Algebra (Part I).

1. $R(A)$ = column space of A; dimension r. (def).
2. $N(A)$ = null space of A; dimension $n-r$
3. $R(A^t)$ = row space of A; dimension r.
4. $N(A^t)$ = left-null space of A; dimension $m-r$.

Example: $A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$ rank 1. $r = 1$

$m = n = 2$.

1. The column space $= \{ c \begin{pmatrix} 1 \\ 3 \end{pmatrix} / c \in \mathbb{R} \}$
2. the null space $= \{ c \begin{pmatrix} -2 \\ 1 \end{pmatrix} / c \in \mathbb{R} \}$
3. The row space $= \{ c \begin{pmatrix} 1 \\ 2 \end{pmatrix} / c \in \mathbb{R} \}$
4. The left-null space $= \{ c \begin{pmatrix} -3 \\ 1 \end{pmatrix} / c \in \mathbb{R} \}$

Every matrix of rank one has the form

$$A = u v^t.$$

i.e. the rows are multiples of the same vector v^t, and the columns are multiples of the same vector u.

LINEAR TRANSFORMATIONS.

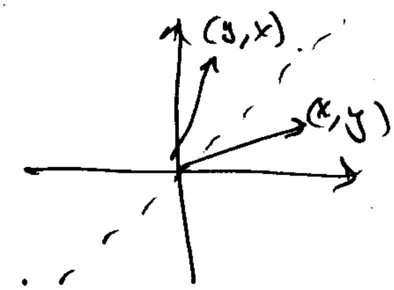
1. Stretching $A = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$ takes $x \longrightarrow cx$:

$$Ax = cx.$$

2. Rotation. $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

A takes the vector x, and rotates by θ degrees.

3. Reflection: $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.



A Takes $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ into $\longrightarrow \begin{pmatrix} y \\ x \end{pmatrix}$.

4. A projection: A is a projection matrix if it takes the whole space onto a lower-dimensional subspace. $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

IMPORTANT FACT: Some transformations are not possible with matrices.

(a) It is impossible to move the origin:

$$A \cdot \vec{0} = \vec{0}, \text{ for all matrices.}$$

(b) If $\vec{x} \longrightarrow \vec{x}'$, then $2\vec{x} \longrightarrow 2\vec{x}'$ (and no other).

In general $\vec{x} \longrightarrow \vec{x}' \Rightarrow c\vec{x} \longrightarrow c\vec{x}'$,

since $$A(c\vec{x}) = c(Ax).$$

(c) If $x \longrightarrow x'$, $y \longrightarrow y'$ then $x+y$ exactly goes to $x'+y'$, and no other,

since $$A(x+y) = Ax + Ay.$$

<u>Def</u>. Transformation that obey (a), (b), (c) are called <u>linear transformations</u>

For all c, d numbers, and x, y vectors, matrix multiplication satisfy:

$$A(cx + dy) = c(Ax) + d(Ay)$$

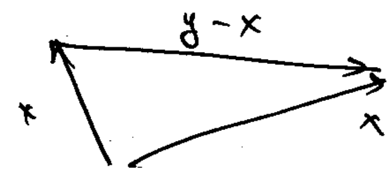
Every trasformation that meets this requirement is a <u>linear transformation</u>.

Orthogonality.

The length of a vector x in $\mathbb{R}^n$ is denoted by $\|x\|$ and is computed by
$$\|x\| = \sqrt{x_1^2 + \cdots + x_n^2},$$
where $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.

Notice that $\|x\|^2 = x^t x$.

Pythagoras formula.



The vectors x, y are orthogonal if $\|x\|^2 + \|y\|^2 = \|x-y\|^2$,
This implies
$$\boxed{x^t y = 0}$$

Def. The quantity $x^t y$ is the inner product of the vectors x and y in $\mathbb{R}^n$. It is zero iff x and y are orthogonal.

Orthogonality of two subspaces. This requires every vector in one subspace to be orthogonal to every vector in the other.

Def. Two subspaces V and W of $\mathbb{R}^n$ are orthogonal if $v^t w = 0$, $\forall v \in V$ and $\forall w \in W$

$\boxed{\begin{array}{l}\underline{\text{Theorem}} \quad A \text{ is } m\times n \text{ matrix.} \\ \text{The row space is orthogonal to the null space (in } \mathbb{R}^n) \\ \qquad R(A^t) \perp N(A) \\ \text{The column space is orthogonal to the left null space} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{(in } \mathbb{R}^m) \\ \qquad R(A) \perp N(A^t)\end{array}}$

<u>Def</u> Given a subspace V of $\mathbb{R}^n$, the space of all vectors orthogonal to V is called the <u>orthogonal complement of V</u>, and denoted by $V^\perp$ (V-orthogonal <u>or</u> V-perp).

Now, remember

$N(A)$ and $R(A^t)$ are subspaces of $\mathbb{R}^n$

$N(A^t)$ and $R(A)$ are subspaces of $\mathbb{R}^m$

If fact, we can prove that:

$$\boxed{\begin{array}{l} N(A) \oplus R(A^t) = \mathbb{R}^n \\ N(A^t) \oplus R(A) = \mathbb{R}^m \end{array}}$$

Furthermore: 04070208.

$\boxed{\begin{array}{l}\text{The Fundamental Theorem of Linear Algebra: Part II}\\ \text{(a) The null space is the orthogonal complement of the row space in } \mathbb{R}^n:\ \boxed{N(A) = (R(A^t))^\perp}\\ \text{(b) The left null space is the orthogonal complement of the column space. in } \mathbb{R}^m:\ \boxed{N(A^t) = (R(A))^\perp}\end{array}}$

This tells us what systems of equations can be solved!

Let's do this as follows. Look at:

$$Ax = b.$$

This equation says

b is in the column space of A.

Therefore, by (b),

b is orthogonal to the left null space of A,

i.e., $\forall y \in \mathbb{R}^m$ s.t. $A^t y = 0$, then $b^t y = 0$.

The converse is also true.

This proves the following Theorem.

Theorem. The equation $Ax = b$ is solvable

if and only if

whenever $A^t y = 0$, then $b^t y = 0$.

From this, the Fredholm alternative follows:

<u>The Fredholm alternative</u>. For any A and b, one and only one, of the following systems has a solution:

(1) $Ax = b$

(2) $A^t y = 0, \quad y^t b \neq 0$.

In other words, either b is in the column space, $R(A)$, of A, or there is $y_0 \in N(A^t)$, such that y_0 and b are orthogonal.

Determinants 04070208

$$\det : \mathbb{R}^{n\times n} \longmapsto \mathbb{R}$$

or

$$\det : \mathbb{C}^{n\times n} \longmapsto \mathbb{C}.$$

$\llcorner$ square matrices.

The determinant is a function that takes the space of square matrices into the corresponding field ($\mathbb{R}$ or $\mathbb{C}$).

Why determinants are important?

(1) Test for invertibility.
If the determinant of A is zero, then A is singular,
If $\det A \neq 0$, then A is invertible.

(2) The determinant measures the volume of a parallelepiped P in an n-dimensional space.

(3) The determinant gives formula for pivots (Gaussian elimination)

$$\det(A) = \pm(\text{product of pivots})$$

(4) The det measures the dependence of $A^{-1}b$ on each element of b. $A^{-1}b$

If b is changed in experiment, then $x = A^{-1}b$ is given by determinants.

Examples: $A = (a)$: $\det A = 0$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \det A = ad - bc$$

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} : \quad \det A = a \det\begin{pmatrix} e & f \\ h & k \end{pmatrix} - b \det\begin{pmatrix} d & f \\ g & k \end{pmatrix} + c \det\begin{pmatrix} d & e \\ g & h \end{pmatrix}$$

~~Properties~~. Notation.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$|A| = \det A.$$

The simple thing about determinants are not the explicit formulas, but the properties it possesses.

Properties. 1) $\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$ ~~and~~ $\cancel{\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$

1) The determinant depends linearly on the first row:

$$\begin{vmatrix} \alpha a + \beta a' & \alpha b + \beta b' \\ c & d \end{vmatrix} = \alpha \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \beta \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}.$$

2) The determinant change sign when two rows are changed: $\begin{vmatrix} c & d \\ a & b \end{vmatrix} = - \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

Corollary The determinant depends linearly on each row separately

3) The determinant of the identity $I_{n\times n}$ is 1.

$$\det(1) = 1, \quad \det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1, \quad \det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1.$$

4) If two rows of A are equal, then $\det A = 0$.

$$\begin{vmatrix} a & b \\ a & b \end{vmatrix} = ab - ba = 0.$$

(It follows from 2)).

5) The elementary operation of substraction a multiple of a row from another row leave the determinant unchanged.

$$\begin{vmatrix} a-lc & b-ld \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} - l\begin{vmatrix} c & d \\ c & d \end{vmatrix}$$

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix},$$

by rules (1) and (4).

6) If A has a zero row, then $\det A = 0$.

$$\begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} = 0$$

7) If A is triangular, then:

$$\det A = a_{11} a_{22} a_{33} \cdots a_{nn}$$

$$A = \begin{pmatrix} a_{11} & & * \\ & a_{22} & \\ 0 & & \ddots \\ & & & a_{nn} \end{pmatrix}.$$

8) If A is singular, then $\det A = 0$.
If A is invertible, then $\det A \neq 0$.

9) For any two matrices $A, B \in \mathbb{R}^{n\times n}$, the determinant of the product AB is the product of the determinants of each matrices.

$$\det(AB) = (\det A)(\det B)$$

10) The transpose of A has the same determinant as A.

$$\det(A^t) = \det A.$$

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EIGENVALUES AND EIGENVECTORS

Dynamical systems
Normal modes

Our main goal to use Eigenvalues and Eigenvectors is to solve differential equations.

Find $v = v(t)$, $w = w(t)$ that satisfy:

$$(*) \quad \frac{dv}{dt} = 4v - 5w \qquad v(0) = 8 \quad (*)$$
$$\frac{dw}{dt} = 2v - 3w \qquad w(0) = 5$$

If we define: the vector $u = \begin{pmatrix} v \\ w \end{pmatrix}$, the ms

$$u = \begin{pmatrix} v \\ w \end{pmatrix}, \quad u_0 = \begin{pmatrix} v(0) \\ w(0) \end{pmatrix}, \quad A = \begin{pmatrix} 4 & -5 \\ 2 & -3 \end{pmatrix}$$

then, system $(*)$ becomes:

$$\frac{du}{dt} = Au, \quad u(0) = u_0. \quad (**)$$

$(**)$ How to find solutions?

If A is 1×1: $A = (a)$,

$$\frac{du}{dt} = au, \quad u(0) = u_0,$$

then: $\boxed{u(t) = e^{at} u_0}$

If $a = \alpha + i\beta$ is complex, The solution is:

(a) $\alpha > 0$, unstable, approaches ∞
(b) $\alpha = 0$, neutrally stable, remains bounded
(c) $\alpha < 0$, stable, goes to zero

} even if they oscillate due to $e^{i\beta t}$.

Look for solution of the system of the form.

$$v(t) = e^{\lambda t} y$$
$$w(t) = e^{\lambda t} z$$

Same factor ; constants

Define: $x = \begin{pmatrix} y \\ z \end{pmatrix}$. Then

$$u(t) = e^{\lambda t} x$$

Then substitute into system (*) (or (**)):

$$\lambda e^{\lambda t} y = 4 e^{\lambda t} y - 5 e^{\lambda t} z$$
$$\lambda e^{\lambda t} z = 2 e^{\lambda t} y - 3 e^{\lambda t} z$$

i.e.

$$4y - 5z = \lambda y$$
$$2y - 3z = \lambda z.$$

or, in matrix form:

$$\lambda e^{\lambda t} x = A e^{\lambda t} x \quad \Rightarrow \quad \boxed{Ax = \lambda x}$$

Fundamental equation for eivals and eivct's.

$\lambda \equiv$ eigenvalue of A,
$x =$ the associated eigenvector of A

We can also write: 004070208.

$Ax = \lambda I x$, where I = identity matrix.

i.e.

$$\boxed{(A - \lambda I)x = 0.} \qquad (***)$$

- The vector x is in the nullspace of $A - \lambda I$.
- The number λ is chosen s.t. $A - \lambda I$ has a nullspace.

Obviously, $x = 0$ is in the nullspace of $A - \lambda I$, but this does not gives information for the system of differential equation ($u(t) = e^{\lambda t} x = 0$, but does not satisfy $u(0) = u_0$).

Look for solutions of $(***)$ s.t. $x \neq 0$, i.e., the matrix $A - \lambda I$ must be singular, i.e.

The number λ is an eigenvalue of A, iff

$$\det(A - \lambda I) = 0. \quad \text{(Characteristic equation)}$$

and each solution, λ, has a corresponding eigenvector $x \neq 0$,

$$(A - \lambda I)x = 0, \text{ i.e. } Ax = \lambda x$$

$p(\lambda) = \det(A - \lambda I)$: characteristic polynomial

its zeroes, $p(\lambda) = 0$, are the eigenvalues.

For our example.

$$A - \lambda I = \begin{pmatrix} 4-\lambda & -5 \\ 2 & -3-\lambda \end{pmatrix}$$

$$p(\lambda) = \det(A - \lambda I) = (4-\lambda)(-3-\lambda) + 10$$

i.e. $\det(A - \lambda I) = \lambda^2 - \lambda - 2$.

The eigenvalues solve

$$\det(A - \lambda I) = 0$$

$$\lambda^2 - \lambda - 2 = 0$$

$$\Rightarrow \quad (\lambda + 1)(\lambda - 2) = 0$$

Eigenvalues $\boxed{\begin{aligned} \lambda_1 &= -1 \\ \lambda_2 &= 2 \end{aligned}}$

For each λ_1, λ_2, we have one x_1, and x_2.

$$\lambda_1 = -1; \quad (A - \lambda_1 I)x_1 = \begin{pmatrix} 5 & -5 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \boxed{x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

$$\lambda_2 = 2; \quad (A - \lambda_2 I)x_2 = \begin{pmatrix} 2 & -5 \\ 2 & -5 \end{pmatrix}\begin{pmatrix} y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \boxed{x_2 = \begin{pmatrix} 5 \\ 2 \end{pmatrix}}$$

Note that αx_1 is also eigenvector corresponding to λ_1, for any $\alpha \in \mathbb{R}$, $\alpha \neq 0$. Similarly, αx_2 is eivct for λ_2.

Steps in solving the eigenvalue problem. 04070208.

1. Compute the determinant of $A-\lambda I$
2. Find the roots of this polynomial
3. For each eigenvalue, solve the equation $(A-\lambda I)x=0$

Go back to the diff. eq. The pure exponential solutions are:

$$u_1 = e^{\lambda_1 t}x_1 = e^{-t}\begin{pmatrix}1\\1\end{pmatrix} \text{ and } u_2 = e^{\lambda_2 t}x_2 = e^{2t}\begin{pmatrix}5\\2\end{pmatrix}.$$

The general solution is a combination of both.

$$u(t) = c_1 u_1(t) + c_2 u_2(t).$$

i.e

$$u(t) = c_1 e^{\lambda_1 t}x_1 + c_2 e^{\lambda_2 t}x_2.$$

We have to adjust c_1 and c_2 to $u_0 = (8,5)^t$, the initial condition.

$$c_1x_1 + c_2x_2 = u_0 \Rightarrow \begin{pmatrix}1 & 5\\1 & 2\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}8\\5\end{pmatrix}$$

$\Rightarrow \begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}3\\1\end{pmatrix}$. Then, the solution to the initial value problem (*) is:

$$\boxed{u(t) = 3e^{-t}\begin{pmatrix}1\\1\end{pmatrix} + e^{2t}\begin{pmatrix}5\\2\end{pmatrix}}$$

or $v(t) = 3e^{-t} + 5e^{2t}$ and $w(t) = 3e^{-t} + 2e^{2t}$

If a matrix is diagonal or triangular.

$$A = \begin{pmatrix} a_{11} & & 0 \\ & a_{22} \ddots & \\ 0 & & a_{nn} \end{pmatrix}; \quad A = \begin{pmatrix} a_{11} & & * \\ & a_{22} \ddots & \\ & & a_{nn} \end{pmatrix},$$

its eivals are $\lambda_1 = a_{11}, \lambda_2 = a_{22}, \cdots, \lambda_n = a_{nn}$.

<u>Def</u> The <u>trace</u> of A is defined by

$$tr(A) = a_{11} + a_{22} + \cdots + a_{nn}.$$

<u>Thm</u>

$$tr\, A = \lambda_1 + \lambda_2 + \cdots + \lambda_n$$

$$\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$$

The Diagonal form of a Matrix.

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Suppose A is an $n \times n$ matrix, with n linearly independent eivectors, $p_1, p_2, \cdots, p_n$. Choose

$$P = (p_1, p_2, \cdots, p_n).$$

Then, $P^{-1}AP$ is a diagonal matrix Λ, with the eigenvalues along the diagonal:

$$P^{-1}AP = \Lambda = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}.$$

The eivtr p_j corresponds to the eigenvalue λ_j, $j = 1, 2, \cdots, n$.

P = eigenvector matrix

Λ = eigenvalue matrix.

Similarly:

$$A = P \Lambda P^{-1}$$

Proof:

$$AP = A(p_1, \cdots, p_n) = (Ap_1, \cdots, Ap_n) =$$

$$= (\lambda_1 p_1, \cdots, \lambda_n p_n) = (p_1, \cdots, p_n) \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

$$\Rightarrow \quad AP = P\Lambda$$

Then: $\boxed{A = P \Lambda P^{-1}}$, i.e. $\boxed{P^{-1}AP = \Lambda}$

Rmk 1 Any matrix with distinct eivals is diagonalizable.

Rmk 2 The eivct' matrix P is not unique.
(since p_1 and αp_1 are eivct's, $\alpha \neq 0$)

Rmk 3 Any other matrices S' will not produce a diagonal Λ.

$$AS' \neq S'\Lambda.$$

Rmk 4 Not all matrices are diagonalizable.

Example. $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Eivals $\lambda = 0, 0$. (its algebraic multiplicity is 2),

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \text{ only.}$$

The eival $\lambda = 0$ (even it is a double eival, multiplicity 2), it has a one-dimensional space of eivectors.
I.e., the geometric multiplicity is 1, there is only one independent eivct, and we cannot construct P.

Diagonalizability is concerned with eigenvectors
Invertibility is concerned with eigenvalues

Thm If the eivectors $p_1, p_2, \dots, p_n$ correspond to different eivals $\lambda_1, \lambda_2, \dots, \lambda_n$, then the eigenvectors are linearly independent.

Proof: Consider

$$c_1 p_1 + c_2 p_2 = 0.$$

We must force $c_1 = 0$, and $c_2 = 0$.

$$A \mid c_1 p_1 + c_2 p_2 = 0$$

$$\lambda_2 \mid c_1 p_1 + c_2 p_2 = 0$$

imply

$$c_1 A p_1 + c_2 A p_2 = c_1 \lambda_1 p_1 + c_2 \lambda_2 p_2 = 0$$

$$c_1 \lambda_2 p_1 + c_2 \lambda_2 p_2 = 0$$

Substract:

$$c_1 (\lambda_1 - \lambda_2) p_1 = 0.$$

Since $\lambda_1 \neq \lambda_2$, $p_1 \neq 0$, then $c_1 = 0$.
Similarly $c_2 = 0$. An inductive argument shows that $p_1, p_2, \cdots, p_n$ are linearly independent

QED.

Powers and Products: A^k and AB

Consider A has eivals $\lambda_1, \lambda_2, \ldots, \lambda_n$ w/ corresponding eivectors $p_1, p_2, \cdots, p_n$. Then,
the eigenvalues of A^k are $\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k$ with same eigenvectors $p_1, p_2, \cdots, p_n$, and:

$$A^k = P \Lambda^k P^{-1}$$

If λ_j is eival with eivct p_j, of A:

$$A p_j = \lambda_j p_j .$$

Then $$A^2 p_j = \lambda_j A p_j = \lambda_j^2 p_j$$

Then λ_j^2 is eival of A^2, w/ eivct p_j

Similarly, $$A^k p_j = \lambda_j^k p_j ,$$

and the eival. matrix is:

$$\begin{pmatrix} \lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_n^k \end{pmatrix} = \Lambda^k$$

and eivct matrix:

$$P = (p_1, \cdots, p_n) .$$

Then:

$$A^k = P \Lambda^k P^{-1},$$

as we can easily verify:

$$A^k = \underbrace{(P \Lambda P^{-1})(P \Lambda P^{-1}) \cdots (P \Lambda P^{-1})(P \Lambda P^{-1})}_{k \text{ times}}$$

$$= P(\Lambda \Lambda \cdots \Lambda \Lambda)P^{-1}$$

$$= P \Lambda^k P^{-1}$$

Thm. Assume A and B are diagonalizable. 04070208.
A and B share the same eivect's iff $AB = BA$.

Proof. If they share the same eivect's, then P diagonalizes both A and B:

$$A = P\Lambda_1 P^{-1} \qquad B = P\Lambda_2 P^{-1}$$

Now:

$$AB = (P\Lambda_1 P^{-1})(P\Lambda_2 P^{-1}) = P\Lambda_1\Lambda_2 P^{-1}$$
$$= P\Lambda_2\Lambda_1 P^{-1} = (P\Lambda_2 P^{-1})(P\Lambda_1 P^{-1})$$
$$= BA.$$

Conversely, if $AB = BA$, and assume the eivals of A are different. Then:

From $Ax = \lambda x$, it follows $ABx = BAx = B\lambda x =$
$= \lambda Bx$.

Then, Bx is eivct, with same eival λ. Since the eivals are different, then the eivct's should be linearly indep. This force Bx and x to be proportional, i.e.,

$$Bx = \mu x,$$

and they share the same eivecters. If the eigenvalues are repeated, the proof is longer.

Similarly $ABx = A\mu x = \mu Ax = \mu\lambda x,$
and $\mu\lambda$ is eival of AB.

DIFFERENCE EQUATIONS.

They move forward in a finite number of steps.
(differential equations do it continuously).

Suppose you invest \$ 1000 at rate 6%.

If compounded once a year:

Difference Equation.

$$P_{k+1} = (1 + 0.06) P_k \quad \text{or} \quad \boxed{P_{k+1} = 1.06 P_k}$$

After 5 years:

$$P_5 = (1+0.06)^5 P_0 = \$1338$$

If the compound is ~~mon~~ reduced to each month:

$$P_{k+1} = \left(1 + \frac{0.06}{12}\right) P_k$$

After 5 years = 60 months:

$$P_{60} = \left(1 + \frac{0.06}{12}\right)^{5\cdot 12} P_0 = \left(1 + \frac{0.06}{12}\right)^{60} P_0$$

$$= \$ 1349.$$

If daily compounded:

$$P_{k+1} = \left(1 + \frac{0.06}{365}\right)^{5\cdot 365} P_0 = \$1349.83$$

If compound N times a year:

$$P_{k+1} = \left(1 + \frac{0.06}{N}\right) P_k$$

After one year

$$P_N = \left(1 + \frac{0.06}{N}\right)^N P_0 .$$

After 5 years

$$P_{5N} = \left(1 + \frac{0.06}{N}\right)^{5 \cdot N} P_0 .$$

Now, if we do it continuously, $N \longrightarrow \infty$.. Then:

$$\left(1 + \frac{0.6}{N}\right)^{5 \cdot N} = \left(1 + \frac{1}{\left(\frac{N}{0.06}\right)}\right)^{5 \cdot N} \underset{M = \frac{N}{0.06}}{=} \left(1 + \frac{1}{M}\right)^{(M)(0.3)} \longrightarrow$$

$$\xrightarrow[\substack{N \to \infty \\ \text{i.e. } M \to \infty}]{} e^{0.3} .$$

Then: $P_{5N} = \left(1 + \frac{0.06}{N}\right)^{5 \cdot N} P_0 \longrightarrow e^{0.3} \cdot 1000 \simeq$

$\simeq \$1349.87$

One can switch to a differential equation:

From: $P_{k+1} = \left(1 + \frac{0.06}{N}\right) P_k = \left(1 + 0.06 \Delta t\right) P_k,$

where Δt are the increments in time of compounded interest:

$$P_{k+1} - P_k = 0.06 \Delta t \, P_k \Rightarrow \frac{P_{k+1} - P_k}{\Delta t} = 0.06 \, P_k$$

If $\Delta t \longrightarrow 0$:

$$\frac{dp}{dt} = 0.06\, p.$$

Then: $\boxed{p(t) = e^{0.06t}\, p_0.}$

After 5 years: $p(5) = e^{0.3} \cdot 1000 = \$1349.87$

same

The Fibonacci sequence.

$$0, 1, 1, 2, 3, 5, 8, 13, 21, \dots$$

$$\boxed{F_{k+2} = F_{k+1} + F_k,} \qquad F_0 = 0, \quad F_1 = 1.$$

Refine: this sequence as a system:

$$\begin{aligned} F_{k+2} &= F_{k+1} + F_k \\ F_{k+1} &= F_{k+1} \end{aligned} \Rightarrow \begin{pmatrix} F_{k+2} \\ F_{k+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{k+1} \\ F_k \end{pmatrix}.$$

or $\quad u_{k+1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} u_k$

$\boxed{u_{k+1} = A u_k}$ Difference equation

The solution to $u_{k+1} = A u_k$:

is $\boxed{u_k = A^k u_0}$

If A^k can be diagonalized; $A = P \Lambda P^{-1}$,

then $$u_k = A^k u_0 = P \Lambda^k P^{-1} u_0.$$

$$= (P_1 \cdots P_n) \begin{pmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{pmatrix} P^{-1} u_0.$$

If the initial condition is written in terms of: $P_1, \cdots, P_n$:

$$u_0 = c_1 P_1 + \cdots + c_n P_n, \quad \cdots\cdots \quad (*)$$

then: $$u_k = c_1 \lambda_1^k P_1 + \cdots + c_n \lambda_n^k P_n, \quad \cdots\cdots (**).$$

i.e., the solution is a combination of the pure solutions:

$$\lambda_j^k P_j.$$

The role of the c's is to match the initial condition $(*)$.

$$u_0 = (P_1, \cdots, P_n) \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = P c.$$

i.e.

$$P^{-1} u_0 = c,$$

and

$$u_k = P \Lambda^k P^{-1} u_0 = P \Lambda^k c, = (P_1 \cdots P_n) \begin{pmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{pmatrix} c,$$

which turns to be $(**)$ above.

Stability of solutions of difference equations

Thm. The difference equations $u_{k+1} = A u_k$ has solutions which are

(a) stable, if all eigenval's $|\lambda_k| < 1$.

(b) neutrally stable, if some eivals are $|\lambda_k| = 1$ and the rest $|\lambda_k| < 1$.

(c) unstable if at least one eival $|\lambda_k| > 1$.

In the stable case, the powers A^k approach to zero, and so does the solutions $u_k = A^k u_0$

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Applications of Linear Algebra to Differential Equations and the Exponential e^{At}.

For difference equations $u_{k+1} = A u_k$ the solutions are

$$u_k = A^k u_0.$$

For differential equation, $\frac{du}{dt} = Au$, similarly, we have:

$$u(t) = e^{At} u_0,$$

where it depends on the exponential of A, e^A.

We are going to define e^A in this section.

Consider

$$\frac{du}{dt} = Au = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} u.$$

1st step Find eivals and eivectors.

$$\det(A - \lambda I) = \begin{pmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{pmatrix} = (\lambda+2)^2 - 1 = 0$$

i.e.

$$\lambda^2 + 4\lambda + 3 = 0$$
$$(\lambda+3)(\lambda+1) = 0$$

$\boxed{\begin{matrix} \lambda_1 = -1 \\ \lambda_2 = -3 \end{matrix}}$ are the eivals.

$$\lambda_1 = -1, \quad (A - \lambda_1 I)x_1 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \boxed{x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

$\lambda_2 = -3$, $(A - \lambda_2 I)x_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} = 0 \Rightarrow \boxed{x_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$

The pure exponential solutions are

$$u_1(t) = e^{\lambda_1 t} x_1 \qquad u_2(t) = e^{\lambda_2 t} x_2.$$

The general solution, is any linear combination of both:

$$u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2$$

$$= (x_1, x_2)\begin{pmatrix} c_1 e^{\lambda_1 t} & 0 \\ 0 & c_2 e^{\lambda_2 t} \end{pmatrix}$$

$$= (x_1, x_2)\begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

At $t=0$:

$$u_0 = c_1 x_1 + c_2 x_2 = (x_1\, x_2)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$= P\, c.$$

$$\Rightarrow \quad c = P^{-1} u_0$$

Then:

$$u(t) = P\begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} c = P\begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} P^{-1} u_0.$$

Compare with $u(t) = e^{A\cdot t} u_0$.

Then: 04070208.

$$e^{At} = P\begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t}\end{pmatrix} P^{-1}.$$

or:

$$(\star) \cdots\cdots e^{At} = P\, e^{\Lambda t} P^{-1} \cdots\cdots (\star)$$

The fundamental formula:

$$\boxed{u(t) = P e^{\Lambda t} S^{-1} u_0}$$

The key matrices:

$$\boxed{\Lambda = \begin{pmatrix} -1 & 0 \\ 0 & -3\end{pmatrix} \quad \text{and} \quad e^{\Lambda t} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-3t}\end{pmatrix}}$$

Define $e^{\mathbf{A}t}$:

As we have $e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$,

we define:

$$e^{At} = I + At + \frac{A^2}{2!}t^2 + \cdots + \frac{A^n}{n!}t^n + \cdots$$

This series is convergent, and has the properties:

$$e^{At}e^{As} = e^{A(t+s)}$$

$$e^{At}e^{-At} = I$$

$$\frac{d}{dt}\left(e^{At}\right) = A e^{At}.$$

We see then, that $u(t) = e^{At}u_0$, solves the diff. eq. $\frac{du}{dt} = Au$.

This solution must be the same as the one previously found: $u = Pe^{\Lambda t}P^{-1}u_0$.

$$e^{At} = I + At + \frac{A^2 t}{2!} + \cdots + \frac{A^n t^n}{n!} + \cdots$$

$$= I + P\Lambda P^{-1}t + \frac{P\Lambda^2 P^{-1}}{2!}t^2 + \cdots + \frac{P\Lambda^n P^{-1}}{n!}t^n + \cdots$$

$$= P\left(I + \Lambda t + \frac{\Lambda^2}{2!}t^2 + \cdots + \frac{\Lambda^n}{n!}t^n + \cdots\right)P^{-1}$$

$$= Pe^{\Lambda t}P^{-1}$$

The two different notations coincide.

<u>Example:</u> the exponential of $At = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}t$ is:

$$e^{At} = Pe^{\Lambda t}P^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} e^{-t} & e^{-t} \\ e^{-3t} & -e^{-3t} \end{pmatrix}$$

$$e^{At} = \frac{1}{2}\begin{pmatrix} e^{-t} + e^{-3t} & e^{-t} - e^{-3t} \\ e^{-t} - e^{-3t} & e^{-t} + e^{-3t} \end{pmatrix}.$$

Notice that, at $t=0$,

$$e^{A\cdot 0} = \frac{1}{2}\begin{pmatrix} 1+1 & 1-1 \\ 1-1 & 1+1 \end{pmatrix} = I,$$

is the identity matrix.

We also called e^{At}, the fundamental matrix.

If A can be diagonalized $A = P\Lambda P^{-1}$, then the solution to the differential equation $\frac{du}{dt} = Au$ is given by:

$$u(t) = e^{At}u_0 = P\, e^{\Lambda t} P^{-1} u_0$$

The columns of P are the eigenvectors of A, so that:

$$u(t) = (P_1 \cdots P_n)\begin{pmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{pmatrix} S^{-1} u_0$$

$$= c_1 e^{\lambda_1 t} P_1 + \cdots + c_n e^{\lambda_n t} P_n$$

is the general solution, combination of pure exponentials.

The constants c_i's, match the initial condition u_0:

$$c = P^{-1}u_0 .$$

* The matrix e^{At} is <u>never</u> <u>singular</u>:

$$\det e^{At} = e^{\lambda_1 t} e^{\lambda_2 t} \cdots e^{\lambda_n t} = e^{(\lambda_1+\lambda_2+\cdots+\lambda_n)t}$$

$$= e^{tr(A)t} \neq 0.$$

* If n vectors $v_1, v_2, \ldots, v_n$ are linearly independent, and are chosen as initial conditions, they will remain linearly independent.

These vectors will evolve as:

$$\left(e^{At}v_1, e^{At}v_2, \ldots, e^{At}v_n\right) = e^{At}\left(v_1, v_2, \ldots, v_n\right)$$

Now:

$$\det\left(e^{At}v_1, \ldots, e^{At}v_n\right) = \underbrace{\det(e^{At})}_{\neq 0} \; \underbrace{\det(v_1, v_2, \ldots, v_n)}_{\neq 0}.$$

$\neq 0$, since v_j's are linearly independent.

Then, $e^{At}v_1, \ldots, e^{At}v_n$ are lin. independent

Def: $W\left(v_1(t), v_2(t), \ldots, v_n(t)\right) \equiv \det\left(v_1(t), \ldots, v_n(t)\right)$, is called the Wronskian of $v_1(t), v_2(t), \ldots, v_n(t)$.

Stability of solutions.

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We established that a solution of the diff.eq. $\frac{du}{dt} = Au$ can be written as:

$$u(t) = Pe^{\Lambda t}P^{-1}u_0 =$$

$$= c_1 e^{\lambda_1 t} p_1 + c_2 e^{\lambda_2 t} p_2 + \dots + c_n e^{\lambda_n t} p_n .$$

i.e., as the linear combination of pure exponential solutions.

The stability is governed by the factors $e^{\lambda_j t}$.

One the real part of λ_j determines the stability, since

$$|e^{\lambda_j t}| = |e^{\alpha_j t + i\beta_j t}| = e^{\alpha_j t}$$

The differential equation $\frac{du}{dt} = Au$ has

- a stable solution if all $\mathrm{Re}\,\lambda_j < 0$ $\left(e^{At} \xrightarrow[t\to\infty]{} 0\right)$
- a neutrally stable solution if $\mathrm{Re}\,\lambda_j \le 0$, and some $\mathrm{Re}\,\lambda_j = 0$. exa
- an unstable solution if $\mathrm{Re}\,\lambda_j > 0$ for at least one eigenvalue