Ecuaciones Diferenciales Ordinarias. UAM-Azcapotzalco.

Examen de Recuperación Trimestre Invierno 2018 | Abril 2018

Answer Key

(1). Solving
$$\frac{dy}{dx} = yx + x\sqrt{y} + \sqrt{x}y + \sqrt{x}\sqrt{y},$$
we observe that: we can write:
$$\frac{dy}{dx} = (x+\sqrt{x})(y+\sqrt{y})$$
which is separable:
$$\int \frac{dy}{y+\sqrt{y}} = \int (x+\sqrt{x})\,dx.$$

Now,
$$\int \frac{1}{y+\sqrt{y}}\,dy = \int \frac{1}{t^2+t}\,2t\,dt = 2\int \frac{1}{t+1}\,dt$$
$$y=t^2,\quad \frac{dy}{dt} = \frac{1}{2\sqrt{y}} = \frac{1}{2t}$$
$$= 2\operatorname{Log}|t+1| = 2\operatorname{Log}(1+\sqrt{y}).$$

Hence:
$$2\operatorname{Log}(1+\sqrt{y}) = \frac{1}{2}x^2 + \frac{2}{3}x^{3/2} + C$$
$$1+\sqrt{y} = C\,e^{\left(\frac{1}{4}x^2 + \frac{1}{3}x^{3/2}\right)}$$
$$\boxed{y(x) = \left(C\,e^{\left(\frac{1}{4}x^2+\frac{1}{3}x^{3/2}\right)} - 1\right)^2}$$

(2) Check if:

$$\left(2y^3 + \tfrac{1}{x}\right) + \left(3xy^2 - \frac{1}{xy}\right)\frac{dy}{dx} = 0$$

is an exact Diff Eq. Let:

$$M(x,y) = 2y^3 + \frac{1}{x} \quad ; \quad N(x,y) = 3xy^2 - \frac{1}{xy}$$

Now:

$$\frac{\partial M}{\partial y} = 6y^2 \quad \text{and} \quad \frac{\partial N}{\partial x} = 3y^2 + \frac{1}{x^2y}$$

Hence, it is <u>not</u> <u>exact</u>. Multiply by the integrating factor $\mu(x,y)$:

$$\mu M(x,y) + \mu N(x,y)\frac{dy}{dx} = 0$$

To be exact, we require:

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N).$$

i.e.

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x$$

If $\mu_y = 0$:

$$\mu_x = \mu\left(\frac{M_y - N_x}{N}\right) = \mu\,\frac{\left(6y^2 - \left(3y^2 + \frac{1}{x^2y}\right)\right)}{\left(3xy^2 - \frac{1}{xy}\right)}$$

$$= \mu\left(\frac{3x^2y^3 - 1}{3x^3y^3 - x}\right) = \frac{\mu}{x}\left(\frac{3x^2y^3 - 1}{3x^2y^3 - 1}\right) = \frac{\mu}{x}$$

Hence:

$$\frac{d\mu}{dx} = \frac{1}{x}\mu \Rightarrow \frac{d}{dx}(\log\mu) = \frac{1}{x} \Rightarrow \log\mu = \log x$$

(where the const. of integ. then $C \equiv 0$)

$\Rightarrow \boxed{\mu(x) = x}$ is the integrating factor.

Then, the Diff. Eq'n becomes:

$$x\left(y^3+\frac{1}{x}\right) + x\left(3xy^2 - \frac{1}{xy}\right)\frac{dy}{dx} = 0$$

is now exact:

$$\underbrace{(2xy^3+1)} + \left(3x^2y^2 - \frac{1}{y}\right)\frac{dy}{dx} = 0,$$

since

$$\tilde{M}(x,y) + \tilde{N}(x,y)\frac{dy}{dx} = 0$$

$$\left.\begin{aligned} \frac{\partial \tilde{M}}{\partial y} &= \frac{\partial}{\partial y}\left(2xy^3+1\right) = 6x^2y^2 \\ \frac{\partial \tilde{N}}{\partial x} &= \frac{\partial}{\partial x}\left(3x^2y^2 - \frac{1}{y}\right) = 6xy^2 \end{aligned}\right\}$$ which coincide.

Then:

$$\left.\begin{aligned} \frac{\partial \Psi}{\partial x} &= 2xy^3 + 1 \\ \frac{\partial \Psi}{\partial y} &= 3x^2y^2 - \frac{1}{y} \end{aligned}\right\} \Rightarrow \begin{aligned} &\Psi(x,y) = x^2y^3 + x + f(y) \\ &\text{and } \Psi(x,y) = x^2y^3 - \log|y| + g(x), \end{aligned}$$

where $f(y)$ and $g(x)$ are functions of just one variable.

Hence: $f(y) = -\log|y|$ and $g(x) = x$, and

$\Psi(x,y) = x^2y^3 + x + \log|y|$, and the solution $\Psi(x,y) = C$

is.

$$\boxed{x^2y^3 + x + \operatorname{Log}|y| = C}$$

which is implicit, since we cannot solve for y (due that we have a polynomial and a transcendental function in y).

(3) The equation can be written as:

$$\frac{dy}{dx} + y = xy^4,$$

which is of the Bernoulli type (with $n = 4$)

Look for $v(x) = y^{\alpha}(x)$. Then, we have to find α.

Hence:

$$\frac{dv}{dx} = \alpha y^{\alpha-1} \frac{dy}{dx} = \alpha y^{\alpha-1}\left(-y + xy^4\right) = -\alpha y^{\alpha} + \alpha x y^{\alpha+3}.$$

Take $\alpha = -3$, so: $\quad \frac{dv}{dx} = +3v - 3x.$

(i.e. $\alpha = 1 - n = 1 - 4 = -3$ ✓). We have the Diff Eq.

$$\frac{dv}{dx} - 3v = -3x.$$

The solution to the homogeneous equation $v_h' - 3v_h = 0$ is $v_h(x) = e^{3x}$

A particular solution has the form:

$$v_p(x) = \alpha x + \beta, \quad \alpha, \beta \text{ - constants to be determined.}$$

Hence:

$$\alpha - 3(\alpha x + \beta) = -3x$$

$$-3\alpha x + (\alpha - 3\beta) = -3x$$

i.e. $\left.\begin{array}{l} -3\alpha = -3 \\ \alpha - 3\beta = 0 \end{array}\right\} \boxed{\begin{array}{l} \alpha = 1 \\ \beta = \frac{1}{3} \end{array}} \Rightarrow v_p(x) = x + \frac{1}{3}$

$$\Rightarrow v(x) = Ce^{3x} + x + \frac{1}{3}.$$

Now $v = y^{-3} \Rightarrow y(x) = v^{-1/3}$

$$\Rightarrow \boxed{y(x) = \frac{1}{\sqrt[3]{Ce^{3x} + x + \frac{1}{3}}}}$$

(4) Because of the unlimited resources, the Malthus model is the appropiate one: $\frac{dP}{dt} = kP$,

whose solution is: $P(t) = P(0)\, e^{kt}$.

If t is in years, and $t=0$ is Jan 1^{st} 2018,

$P(0) = 1000 \Rightarrow \boxed{P(t) = 1000\, e^{kt}}$

We still need to determine k. To do this, we use the fact that:

$$P(3) = 64{,}000 = 2^6 \times 1000.$$

i.e. $1000\, e^{3k} = 2^6 \times 1000$

$\Rightarrow 3e^{3k} = 2^6 \Rightarrow 3k = \log 2^6 \Rightarrow \boxed{k = 2\log 2}$

(a) Hence, the solution is:

$$\boxed{P(t) = 1000\, e^{(2\log 2)t}}$$

or $\boxed{P(t) = 1000\,(4)^t}$

(b) On Jan 1^{st}, 2020, $t = 5$ years. Hence, $P(5) = 1000 \cdot 4^5$

i.e. $P(5) = 1000(2^{10}) \Rightarrow \boxed{P(5) = 1{,}024{,}000 \text{ conejos}}$

(c) We need to find T, such that $P(T) = 256{,}000$.

$\Rightarrow 1000(4^T) = 256{,}000 \Rightarrow 2^{2T} = 2^8$

$\Rightarrow \boxed{T = 4 \text{ years}}$

and this is on $\boxed{\text{January } 1^{st}, 2019}$

(5) We have: $x^2 y'' + 3x y' + y = 0$.

For $y_1(x) = \frac{1}{x}$, we compute $y_1'(x) = -\frac{1}{x^2}$, $y_1'' = \frac{2}{x^3}$. Hence:

$$x^2 y_1'' + 3x y_1' + y_1 = x^2\left(\frac{2}{x^3}\right) + 3x\left(-\frac{1}{x^2}\right) + \frac{1}{x} = \frac{2}{x} - \frac{3}{x} + \frac{1}{x}$$

$= 0$, hence $y_1(x)$ is solution:

To find a second solution, $y_2(x)$, find $v(x)$ such that:

$$y_2(x) = v(x)\, y_1(x).$$

Hence:
$$x^2 y_2'' + 3x y_2' + y_2 = 0$$

implies:
$$x^2\left(v'' y_1 + 2v'y_1' + y_1''\right) + 3x\left(v'y_1 + v y_1'\right) + v y_1 = 0$$

i.e.
$$x^2 y_1 v'' + \left(2x^2 y_1' + 3x y_1\right) v' + v \underbrace{\left(x^2 y_1'' + 3x y_1' + y_1\right)}_{=0} = 0$$

Let $V(x) = v'(x)$, Hence we have the 1st order Diff Eq'n.

$$x^2 y_1 V' + \left(2x^2 y_1' + 3x y_1\right) V = 0.$$

i.e.
$$\frac{x^2}{x} V' + \left(2x^2\left(-\frac{1}{x^2}\right) + 3x\,\frac{1}{x}\right) V = 0$$

ie $\boxed{x V' + V = 0}$ $\Rightarrow \frac{1}{V} V' = \frac{-1}{x} \Rightarrow \text{Log } V = -\text{Log } x$

$$\Rightarrow V(x) = \frac{1}{x} \Rightarrow v' = \frac{1}{x} \Rightarrow v(x) = \text{Log } x$$

Hence, the second solution is $y_2(x) = (\text{Log } x) \cdot \frac{1}{x}$

and the general solution is $\boxed{y(x) = \frac{C_1}{x} + C_2 \frac{\text{Log } x}{x}}$

(6) Here we use the judicious conjecture to solve this Diff. Eq., although variation of parameters also applies.

The Diff. Eq'n is:

$$y'' - 4y' = 12x e^{4x} + 12 \quad \ldots\ldots (\bigstar)$$

(a) We solve the homogeneous Diff. Eq'n

$$Y'' - 4Y' = 0 \quad \ldots\ldots (*)$$

Integrating once:

$$Y' - 4Y = C_1$$

The homogeneous part is:

$$Y_h' - 4Y_h = 0$$

with solution $Y_h(x) = e^{4x}$.

And the particular solution should be a constant:

$$Y_p(x) = C_2.$$

Hence:

$$Y(x) = C_1 e^{4x} + C_2.$$

This solve the homogeneous eq'n $(*)$

This solution can be find also assuming $Y(x) = e^{rx}$, since eq'n $(*)$ is linear, homogeneous and const. coeff. Diff Eq'n. We substitute to get the characteristic eq'n:

$$r^2 - 4r = 0 \quad \text{i.e.}$$

i.e. $r(r-4) = 0$. Then: $r_1 = 0 \Rightarrow y_1(x) = \tilde{C}_1$

$r_2 = 4 \qquad y_2(x) = \tilde{C}_2 e^{4x}$

$$\Rightarrow Y(x) = \tilde{C}_2 e^{4x} + \tilde{C}_1, \text{ same solution}$$

Now, we find a particular solution to eq'n (*).

The first trial is:

$$y_p(x) = (\alpha x+\beta)e^{4x} + \gamma, \quad \left\{ \alpha, \beta, \gamma \text{ constants to be determined} \right.$$

but this repeats the homogeneous solution $Y(x) = C_1 e^{4x} + C_2$,

hence, multiply by x:

$$y_p(x) = (\alpha x^2 + \beta x)e^{4x} + \gamma x.$$

This should work:

$$y_p'(x) = (2\alpha x+\beta)e^{4x} + 4(\alpha x^2 + \beta x^2)e^{4x} + \gamma.$$

$$= \left(4\alpha x^2 + (2\alpha+4\beta)x + \beta\right)e^{4x} + \gamma.$$

$$y_p'' = \left(8\alpha x + (2\alpha+4\beta)\right)e^{4x} + 4\left(4\alpha x^2 + (2\alpha+4\beta)x\right)e^{4x}$$

$$= \left(16\alpha x^2 + (8\alpha + 8\alpha + 16\beta)x + (2\alpha+4\beta) + 4\beta\right)e^{4x}$$

$$= \left(16\alpha x^2 + 16(\alpha+\beta)x + 2(\alpha+4\beta)\right)e^{4x}.$$

Substitute into the Diff. Eq'n.

$$\left(16\alpha x^2 + 16(\alpha+\beta)x + 2(\alpha+4\beta)\right) - 4\left(4\alpha x^2 + 2(\alpha+2\beta)x + \beta\right)e^{4x} - 4\gamma$$

$$= 12xe^{4x} + 12$$

$$\left[(\overbrace{16\alpha - 16\alpha}^{0})x^2 + (16\alpha + \overbrace{\cancel{16\beta} - 8\alpha - \cancel{16\beta}}^{0})x + \left(2(\alpha+4\beta) - 4\beta\right)\right]e^{4x} - 4\gamma$$

$$= -12x e^{4x} + 12.$$

$$\left[(8\alpha)x + (2\alpha+4\beta)\right]e^{4x} - 4\gamma = -12xe^{4x} + 12.$$

Hence, $\left.\begin{aligned} 8\alpha &= -12 \\ 2(\alpha + 2\beta) &= 0 \\ -4\gamma &= 12 \end{aligned}\right\} \Rightarrow \boxed{\begin{aligned} \alpha &= -\frac{3}{2} \\ \beta &= +\frac{3}{4} \\ \gamma &= -3 \end{aligned}}$

Hence, the solution to: (A) is:

$$\boxed{y(x) = C_1 e^{4x} + C_2 + \frac{3}{4}(2x^2 - x)e^{4x} - 3}$$

(7) For the Diff. Eq:

$$y'' + 25y = 3\csc(5x),$$

the Variation of Parameters should be used.

The homogeneous eq'n is:

$$y_h'' = -25y_h,$$

has the solution

$$y_h(x) = C_1\cos(5x) + C_2\sin(5x).$$

Let us call: $y_1(x) = \cos(5x)$ & $y_2(x) = \sin(5x)$.

Then $W[y_1, y_2] = \det\begin{pmatrix} \cos 5x & \sin 5x \\ -5\sin 5x & 5\cos 5x \end{pmatrix} = 5$.

If we call $a(x) = 1$, and $g(x) = 3\csc(5x)$, then, particular solution is: $y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

where

$$A(x) = -\int \frac{y_2(x)g(x)}{a(x)W[y_1,y_2]}\,dx \;;\; B(x) = \int \frac{y_1(x)g(x)\,dx}{a(x)W[y_1,y_2]}$$

$= ? =$

Hence:

$$A(x) = -\int \frac{\sin(5x)\, 3\csc(5x)}{1\cdot 5}\,dx = -\frac{3}{5}\int dx = -\frac{3}{5}x.$$

$$B(x) = \int \frac{\cos(5x)\, 3\csc(5x)}{1\cdot 5}\,dx = \frac{3}{5}\int \frac{\cos(5x)}{\sin(5x)}\,dx$$

$$= \frac{3}{5^2}\operatorname{Log}|\sin(5x)|$$

Then, the general solution is:

$$y(x) = C_1 y_1(x) + C_2 y_2(x) + A(x)\, y_1(x) + B(x)\, y_2(x)$$

i.e.

$$\boxed{y(x) = C_1\cos(5x) + C_2\sin(5x) - \frac{3}{5}x\cos 5x + \frac{3}{25}\sin(5x)\operatorname{Log}|\sin(5x)|}$$

(8) Here, $m = 1/2$ slug, $k = 2$ lb/ft. Then: $b = 0$. and $F_{ext}(t) = 0$.

If we choose the orientation as $\uparrow y$, 0 = equilibrium,

we have $y(0) = -1$ ft

$\dot{y}(0) = 2.$ ft/sec.

(a) The Equation of motion is:

$$m\ddot{y} + ky = 0$$

$$\frac{1}{2}\ddot{y} + 2y = 0$$

$$\ddot{y} + 4y = 0$$

Then

$$\boxed{y(t) = C_1\cos(2t) + C_2\sin(2t)}$$

Using $y(0) = -1$, $\dot{y}(0) = 2$.

$$y(t) = y(0)\cos(2t) + \frac{\dot{y}(0)}{2}\sin(2t)$$

i.e. $\boxed{y(t) = -\cos(2t) + \sin(2t)}$ $(**)$

In alternative form:

$$y(t) = A\cos(2t - \varphi),$$

and we have to find A and φ. Notice that:

$$y(t) = A\cos\varphi\cos(2t) - A\sin\varphi\sin(2t)$$

Comparing with $(**)$,

$$A\cos\varphi = -1$$
$$-A\sin\varphi = 1.$$

Then: $A^2 = A^2(\cos^2\varphi + \sin^2\varphi) = A^2\cos^2\varphi + A^2\sin^2\varphi =$

$$= (-1)^2 + (-1)^2 = 2 \Rightarrow \boxed{A = \sqrt{2}}$$

Also:

$$\tan\varphi = \frac{\sin\varphi}{\cos\varphi} = \frac{A\sin\varphi}{A\cos\varphi} = \frac{-1}{-1} = 1 \Rightarrow \boxed{\varphi = \frac{\pi}{4}}$$

Then:

$$\boxed{y(t) = \sqrt{2}\cos\left(2t - \frac{\pi}{4}\right)}$$

(b) The particle passes through the equilibrium when $y(t) = 0$, i.e. $\sqrt{2}\cos\left(2t - \frac{\pi}{4}\right) = 0$

i.e. (A) $2t_n - \frac{\pi}{4} = (2n+1)\frac{\pi}{2}$, for $n = 0, \pm 1, \pm 2, \dots$ (A)

Now, the velocity is:

$$\dot{y}(t) = -A \sin\left(2t - \frac{\pi}{4}\right)$$

$$= -A \sin\left((2n+1)\frac{\pi}{2}\right)$$

$$= -A(-1)^n$$

$$= A(-1)^{n+1}$$

$\dot{y} < 0$, as the problem requires it. Thus n should be even: $n = 2m$; $m = 0, \pm 1, \pm 2, \pm 3, \dots$

$$\dot{y}(t) = A(-1)^{2m+1} = -A < 0$$

Then, substitute $n = 2m$ in (A) above:

$$2t_m - \frac{\pi}{4} = (4m+1)\frac{\pi}{2}$$

$$t_m - \frac{\pi}{8} = (4m+1)\frac{\pi}{4}$$

$$t_m = m\pi + \frac{\pi}{4} + \frac{\pi}{8} \Rightarrow \boxed{t_m = m\pi + \frac{3\pi}{8}}$$

(c). We must find the times t, when $\dot{y}(t) = -2$ ft/sec.

Then:

$$-A \sin\left(2t - \frac{\pi}{4}\right) = -2$$

$$\sin\left(2t - \frac{\pi}{4}\right) = \frac{2}{A} = \frac{2}{\sqrt{2}} \Rightarrow$$

$\Rightarrow$ The argument should be:

$$2t - \frac{\pi}{4} = \frac{\pi}{4} + 2n\pi$$

or

$$2t - \frac{\pi}{4} = \frac{3\pi}{4} + 2m\pi$$

i.e.

$$2t = \frac{\pi}{2} + 2n\pi$$

or

$$2t = \pi + 2m\pi$$

$$t_n = \frac{\pi}{4} + n\pi = \left(n + \frac{1}{4}\right)\pi$$

or

$$t_m = \pi + m\pi = (m+1)\pi$$

$$\boxed{\begin{array}{ll} t_n = \left(n + \frac{1}{4}\right)\pi, & n = 0, \pm 1, \pm 2, \pm 3, \dots \\ t_m = (m+1)\pi & m = 0, \pm 1, \pm 2, \pm 3, \dots \end{array}}$$

7-bis) Solve the Diff. Eq'n

$$y'' + 25y = \tan(5x)$$

We have to solve first, the homogeneous Diff. Eq'n:

$$y_h'' + 25 y_h = 0$$

$$y_h(x) = C_1 \cos(5x) + C_2 \sin(5x)$$

Now, we use the variation of parameters to find a particular solution of the form:

$$y_p(x) = A(x)\cos(5x) + B(x)\sin(5x),$$

where

$$A(x) = -\int \frac{y_2(x)\,g(x)}{a(x)\,W[y_1,y_2](x)}\,dx \;;\; B(x) = \int \frac{y_1(x)\,g(x)}{a(x)\,W[y_1,y_2](x)}\,dx$$

where $y_1(x) = \cos 5x$

$y_2(x) = \sin(5x)$

$$W[y_1,y_2](x) = \det\begin{pmatrix} \cos 5x & \sin 5x \\ -5\sin 5x & 5\cos 5x \end{pmatrix} = 5$$

$a(x) = 1$ and $g(x) = \tan(5x)$.

Then:

$$A(x) = -\int \frac{\sin(5x)\tan(5x)}{1\cdot 5}\,dx = -\frac{1}{5}\int \frac{\sin^2(5x)}{\cos(5x)}\,dx$$

$$= -\frac{1}{5}\int \frac{1-\cos^2(5x)}{\cos(5x)}\,dx = -\frac{1}{5}\int \frac{1}{\cos 5x}\,dx + \frac{1}{5}\int \cos(5x)\,dx$$

$$= -\frac{1}{5^2}\int \frac{1}{\cos(5x)}\,5\,dx + \frac{1}{5^2}\sin(5x) = -\frac{1}{5^2}\int \sec y\,dy + \frac{1}{5^2}\sin(5x)$$

$$= -\frac{1}{5^2}\operatorname{Log}|\sec y + \tan y| + \frac{1}{5^2}\sin 5x + C.$$

$$= -\frac{1}{5^2}\operatorname{Log}|\sec 5x + \tan 5x| + \frac{1}{5^2}\sin 5x + C.$$

Also.

$$B(x) = \int \frac{\cos(5x)\tan(5x)}{1\cdot 5}\,dx = \frac{1}{5}\int \sin(5x)\,dx = -\frac{1}{5^2}\cos(5x)$$

Then:

$$y_p(x) = \left(-\frac{1}{5^2}\log|\sec(5x)+\tan(5x)| + \frac{1}{5^2}\sin 5x\right)\cos 5x$$

$$+ \left(-\frac{1}{5^2}\cos(5x)\right)\sin(5x)$$

$$= -\frac{1}{25}\log|\sec(5x)+\tan(5x)|\cos 5x$$

Then the general solution is:

$$\boxed{y(x) = C_1\cos(5x) + C_2\sin(5x) - \frac{1}{25}\log|\sec(5x)+\tan(5x)|\cos(5x)}$$

We can also compute.

$$A(x) = -\int \frac{\sin(5x)\tan(5x)}{1\cdot 5}\,dx = -\frac{1}{5}\int \frac{\sin^2 5x}{\cos(5x)}\,dx$$

$$= -\frac{1}{5}\int \frac{\sin^2 5x}{\cos^2(5x)}\cos(5x)\,dx = -\frac{1}{5}\int \frac{\sin^2 5x}{1-\sin^2(5x)}\cos(5x)\,dx$$

$$y = \sin(5x) \ : \ \frac{dy}{dx} = 5\cos(5x)$$

$$= -\frac{1}{5^2}\int \frac{y^2}{1-y^2}\,dy = +\frac{1}{5^2}\int \frac{1-y^2-1}{1-y^2}\,dy =$$

$$= +\frac{1}{5^2}\int 1 - \frac{1}{1-y^2}\,dy = +\frac{1}{5^2}\left[y - \int \frac{1}{1-y^2}\,dy\right]$$

$$= +\frac{1}{5^2}\left[y - \int \frac{1}{2}\frac{1}{1-y} + \frac{1}{2}\frac{1}{1+y}\,dy\right]$$

$$= +\frac{1}{5^2}\left[y - \frac{1}{2}\left(\int\left(\frac{1}{1-y}+\frac{1}{1+y}\right)dy\right)\right]$$

$$= +\frac{1}{S^2}\left[y - \frac{1}{2}\left(-\operatorname{Log}|1-y| + \operatorname{Log}|1+y| \right)\right]$$

$$= +\frac{1}{S^2}\left[y - \frac{1}{2} \operatorname{Log}\left|\frac{1+y}{1-y}\right|\right] = -\frac{1}{S^2}\left(y - \frac{1}{2}\operatorname{Log}\frac{(1+y)^2}{1-y^2}\right)$$

Now, since:

$y = \sin(Sx)$, then,

$$= +\frac{1}{S^2}\left[\sin Sx - \frac{1}{2}\operatorname{Log}\left(\frac{(1+\sin Sx)^2}{1-\sin^2 Sx}\right)\right]$$

$$= +\frac{1}{S^2}\left[\sin Sx - \frac{1}{2}\operatorname{Log}\left(\frac{1+\sin Sx}{\cos Sx}\right)^2\right]$$

$$= +\frac{1}{S^2}\left[\sin(Sx) - \frac{1}{2}\operatorname{Log}\left(\frac{1}{\cos Sx} + \tan Sx\right)^2\right]$$

$$= +\frac{1}{S^2}\left[\sin Sx - \operatorname{Log}\left|\sec(Sx) + \tan(Sx)\right|\right].$$

$$= +\frac{1}{S^2}\sin Sx - \frac{1}{S^2}\operatorname{Log}\left|\sec(Sx) + \tan(Sx)\right|$$

Same result.