UNIVERSIDAD AUTÓNOMA METROPOLITANA - AZCAPOTZALCO
ECUACIONES DIFERENCIALES ORDINARIAS
TRIMESTRE: INVIERNO DE 2018.

EXAMEN # 1. – A

FECHA: VIERNES 16 DE FEBRERO DE 2018

Nombre: SOLUTION KEY

Instrucciones:

- El examen consta de **CINCO** problemas de 20 puntos cada uno.
- Tienen **una** hora con **veinte (20)** minutos para resolverlos.
- Por favor **apaguen sus celulares**. Evíten la pena de quitarles sus exámenes.
- Para recibir puntaje, escriba de forma clara y concisa. Entregue su trabajo limpio y con sus ideas en orden. SIMPLIFIQUE. Muestre sus cuentas. **EXPLIQUE, ARGUMENTE y JUSTIFIQUE** sus respuestas.
- Problema **SIN explicación, desarrollo, justificación o argumento** vale CERO puntos.

PROBLEMAS

(1) (**20 puntos.**) Resuelva la ecuación diferencial

$$\frac{dy}{dt} = \frac{t^4}{\cos y} - t^4 \cos y - \frac{t^6}{\cos y} + t^6 \cos y.$$

(2) (**20 puntos.**) Resuelva la ecuación diferencial e^{4t}

$$e^{2t}\,\frac{dy}{dt} - e^{2t} y = (e^{2t} y)^3.$$

(3) (**20 puntos.**) Resuelva el problema de valores iniciales

$$(\cos t)\frac{dy}{dt} - (\sin t) y = (\sin t),$$
$$y(-\pi) = -2.$$

(4) (**20 puntos.**) Resuelva la ecuación diferencial:

$$\left(\frac{\ln t}{t} - \sin y\right)\frac{dy}{dt} + \left(\frac{\cos y}{t} + \frac{y}{t^2}\right) = 0.$$

(5) (**20 puntos.**) En un caluroso día de verano a 35°C, se encuentra en la Frontera y pide su cerveza bien fría. Entre plática y plática, 10 minutos después de que se la sirven, mide la temperatura de su cerveza y está a 10°C. Otros 5 minutos después, está a 15°C y mejor decide pedir otra. Diga a qué temperatura estaba su cerveza cuando se la sirvieron.

ORDINARY DIFFERENTIAL EQUATIONS | Viernes 19 de octubre de 2018.

Exam #1 (A) SOLUTION KEY

(1) Rewrite the Diff. Eq'n as.

$$\frac{dy}{dt} = \left(t^4 - t^6\right)\left(\frac{1}{\cos y} - \cos y\right)$$

It is separable then.

$$\int \frac{\cos y}{1-\cos^2 y}\, dy = \int (t^4 - t^6)\, dt.$$

i.e.

$$\int \frac{\cos y}{\sin^2 y}\, dy = \frac{1}{5} t^5 - \frac{t^7}{7} + C.$$

i.e.

$$\boxed{\left(\frac{-1}{\sin y}\right) = \frac{1}{5} t^5 - \frac{t^7}{7} + C.}$$

(2) This is a Bernoulli type differential equation.

Then: $v = y^\alpha$. $\quad \frac{dv}{dt} = \alpha y^{\alpha-1} \frac{dy}{dt} = \alpha y^{\alpha-1}\left(e^{2t} y + e^{2t} y^3\right)$

$= \alpha e^{2t} y^\alpha + \alpha e^{2t} y^{\alpha+2}$. Take $\alpha + 2 = 0$ Hence $\alpha = -2$

and $$\frac{dv}{dt} = -2e^{2t} v - 2e^{2t}.$$

we have.

$$\frac{dv}{dt} + 2e^{2t} v = -2e^{2t}$$

is a linear diff eq'n for v.

The integrating factor is:

$$\mu(t) = \exp\left(\int p(t)\,dt\right) = \exp\left(2\int e^{2t}\,dt\right) =$$
$$= \exp\left(e^{2t}\right) = e^{\left(e^{2t}\right)}$$

Now

$$\int q(t)\,\mu(t)\,dt = \int -2e^{2t}\, e^{e^{2t}}\,dt :$$

Let $u = e^{2t}$ Then: $\frac{du}{dt} = 2e^{2t}$. thus:

$$= (-2)\int e^{u}\,\frac{du}{2} = (-2)\frac{e^{u}}{2} = (-2)\frac{e^{\left(e^{2t}\right)}}{2} = -e^{e^{2t}}$$

The solution becomes:

$$v(t) = \frac{1}{\mu(t)}\int \mu(t)q(t)\,dt + \frac{C}{\mu(t)}.$$
$$= \frac{1}{e^{e^{2t}}}\left(-e^{\left(e^{2t}\right)}\right) + \frac{C}{e^{e^{2t}}}$$

$$\boxed{v(t) = -1 + Ce^{-e^{2t}}}$$

But $v(t) = y^{\alpha}(t) \Rightarrow y(t) = (v)^{1/\alpha} = (v)^{-1/2}$

$$\Rightarrow \quad \boxed{y(t) = \frac{1}{\sqrt{-1 + Ce^{-e^{2t}}}}}$$

$= 2 =$

③ $\cos t \frac{dy}{dt} - \sin t\, y = \sin t$ This is a 1st order, linear O.D.E.

In standard form: $\frac{dy}{dt} - \frac{\sin t}{\cos t} y = \frac{\sin t}{\cos t}$.

The integrating factor is:

$$\mu(t) = \exp\left(\int p(t)\,dt\right) = \exp\left(-\int \frac{\sin t}{\cos t}\,dt\right)$$

$$= \exp\left(\log|\cos t|\right) = \cos t$$

thus: $\int \mu(t)\, q(t)\, dt = \int \cos t \frac{\sin t}{\cos t}\, dt = -\cos t$.

Therefore the solution is.

$$y(t) = \frac{1}{\mu(t)} \int \mu(t)\, q(t)\, dt + \frac{C}{\mu(t)} = \frac{1}{\cos t}\left(-\cos t\right) + \frac{C}{\cos t}$$

i.e. $\boxed{y(t) = -1 + \frac{C}{\cos t}}$ is the solution to the ODE.

Alternatively, the original eq'n $(\cos t)\dot{y} - (\sin t) y = \sin t$ is equivalent to: $\frac{d}{dt}\left((\cos t) y\right) = \sin t$. Therefore.

$(\cos t) y(t) = \int \sin t\, dt + C \Rightarrow (\cos t) y(t) = -\cos t + C$

$\Rightarrow \boxed{y(t) = -1 + \frac{C}{\cos t}}$ Same result.

At $t = -\pi$: $y = -2$ Then: $-2 = -1 + \frac{C}{\cos(-\pi)} = -1 + \frac{C}{-1}$

$\Rightarrow C = -1 + 2 = -1$ $\Rightarrow$ The solution to the I.V.P. is $\boxed{y(t) = -1 - \frac{1}{\cos t}}$

= 3 =

(4) $\underbrace{\left(\frac{\text{Log}\, t}{t} - \sin y\right)}_{N(t,y)} \frac{dy}{dt} + \underbrace{\left(\frac{\cos y}{t} + \frac{y}{t^2}\right)}_{M(t,y)} = 0$

Notice that:

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= -\frac{\sin y}{t} + \frac{1}{t^2} \\ \frac{\partial N}{\partial t} &= \frac{1}{t^2} - \frac{\text{Log}\, t}{t^2} \end{aligned}\right\}$$

Not the same.

The DiffEq is <u>not</u> exact.

Now, $\frac{d\mu}{dt} = \left(\frac{M_y - N_t}{N}\right)\mu = \frac{1}{t}\mu$ Then, it works.

$\frac{d\mu}{dt} = \frac{1}{t}\mu \Rightarrow \mu(t) = t$ is the integrating fact

and $(\text{Log}\, t - t \sin y)\frac{dy}{dt} + \left(\cos y + \frac{y}{t}\right) = 0.$

is exact Then:

$\frac{\partial \Psi}{\partial t} = \cos y + \frac{y}{t} \Longrightarrow \Psi(t,y) = t\cos y + y \,\text{Log}\, t + g(y)$

$\Downarrow$

$\frac{\partial \Psi}{\partial y} = \text{Log}\, t - t \sin y \qquad \frac{\partial \Psi}{\partial y} = -t\sin t + \text{Log}\, t + g'($

Comparing $g'(y) = 0 \Rightarrow g(y) = \text{const}$

Then, the solution is: $\boxed{t\cos y + y \,\text{Log}\, t = C}$

(5) The weather is at $35°C = T_w$

We also know $T(10) = 10°C$ with t in minutes.

$$T(15) = 15°C$$

$$T(0) = ?$$

The Diff. Eq'n is.

$$\frac{dT}{dt} = -k(T - T_w).$$

with solution $T(t) = C_1 e^{-kt} + T_w$

From this $\boxed{T(0) = C_1 + T_w}$

We must find C_1 and k:

Notice that
$$Ce^{-10k} + T_w = T(10)$$
$$Ce^{-15k} + T_w = T(15).$$

Then: $Ce^{-10k} = T(10) - T_w$
$Ce^{-15k} = T(15) - T_w$
$\Rightarrow \dfrac{e^{-10k}}{e^{-15k}} = \dfrac{T(10) - T_w}{T(15) - T_w}.$

$\Rightarrow e^{5k} = \dfrac{T(10) - T_w}{T(15) - T_w} \Rightarrow \boxed{k = \frac{1}{5} \operatorname{Log}\left|\frac{T(10) - T_w}{T(15) - T_w}\right|}$

Then: $k = \frac{1}{5} \operatorname{Log}\left|\frac{10 - 35}{15 - 35}\right| = \frac{1}{5} \log\left|\frac{25}{20}\right| \Rightarrow \boxed{k = \frac{1}{5} \log\left(\frac{5}{4}\right) \frac{1}{\text{min}}}$

Now $Ce^{-10k} = T(10) - T_w \Rightarrow C = e^{10k}(T(10) - T_w).$

Since $10k = 2\,\mathrm{Log}\left(\frac{5}{4}\right) = \log\left(\frac{5}{4}\right)^2 = \mathrm{Log}\left(\frac{25}{16}\right)$ $\frac{\text{unit}}{\text{min}}$

Then, $$e^{10k} = \frac{25}{16}.$$

And so: $$C = \frac{25}{16}\left(10 - 35\right) = -\frac{(25)^2}{16} \ (^\circ C)$$

Therefore: $$T(0) = C + T_\omega = -\frac{(25)^2}{16} + 35$$

$$\boxed{T(0) = -4.0625 \ ^\circ C}$$

is the initial temperature.

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EXAMEN # 1. B
FECHA: VIERNES 16 DE FEBRERO DE 2018

Nombre: SOLUTION KEY

Instrucciones:

- El examen consta de **CINCO** problemas de 20 puntos cada uno.
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- Para recibir puntaje, escriba de forma clara y concisa. Entregue su trabajo limpio y con sus ideas en orden. SIMPLIFIQUE. Muestre sus cuentas. **EXPLIQUE, ARGUMENTE y JUSTIFIQUE** sus respuestas.
- Problema **SIN explicación, desarrollo, justificación o argumento** vale **CERO** puntos.

PROBLEMAS

(1) **(20 puntos.)** Resuelva la ecuación diferencial

$$\frac{du}{dx} = xu - \frac{x}{u} - x^3u + \frac{x^3}{u}.$$

(2) **(20 puntos.)** Resuelva la ecuación diferencial

$$\frac{du}{dx} + 2xu = x^2u^2.$$

(3) **(20 puntos.)** Resuelva el problema de valores iniciales

$$x\frac{du}{dx} + (1+x)u = e^{3x}.$$
$$u(1) = 2e^3.$$

(4) **(20 puntos.)** Resuelva la ecuación diferencial:

$$\left(\frac{\sin x}{x} + e^u\right)\frac{du}{dx} + \frac{u\cos x + e^u}{x} = 0.$$

(5) **(20 puntos.)** En un día frío de invierno a 3°C, se prepara su café y se lo sirve. Cinco minutos después, le mide su temperatura y se encuentra a 40°C. Otros 5 minutos después, ya está a 25°C y ya se lo puede tomar. ¿A qué temperatura estaba inicialmente su café cuando se lo sirvió?

ORDINARY DIFFERENTIAL EQUATIONS | Viernes 19 de octubre de 2018

Exam #1 - (B) SOLUTION KEY

(1) We can re-write the Diff. Eq'n as:

$$\frac{du}{dx} = (x - x^3)\left(u - \frac{1}{u}\right).$$

Then, it is separable.

$$\int \frac{1}{u - \frac{1}{u}}\, du = \int (x - x^3)\, dx.$$

$$\int \frac{u}{u^2 - 1}\, du = \frac{x^2}{2} - \frac{x^4}{4} + C_1$$

$$\frac{1}{2} \operatorname{Log}|u^2 - 1| = \frac{x^2}{2} - \frac{x^4}{4} + C_1$$

$$u^2 - 1 = C_2\, e^{x^2 - x^4/2}.$$

$$\boxed{u^2_{(x)} = C_2\, e^{x^2 - x^4/2} + 1}$$

$$\boxed{u(x) = \pm \sqrt{C_2\, e^{x^2 - x^4/2} + 1}}$$

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(2) This is a Bernoulli Diff. Eq.

$v(x) = u^\alpha$ Then: $\frac{dv}{dx} = \alpha u^{\alpha-1} \frac{du}{dx} = \alpha u^{\alpha-1}\left(-2xu + x^2 u^2\right)$

$= -2\alpha x u^\alpha + \alpha x^2 u^{\alpha+1} = -2\alpha x v + \alpha x^2$, if $\alpha + 1 = 0$

then $\alpha = -1$, and we have a linear Diff Eq:

$$\frac{dv}{dx} = 2x v - x^2 \Rightarrow \frac{dv}{dx} - 2xv = -x^2$$

$= 1 =$

The integrating factor is:

$$\mu(x) = \exp\left(\int -2x\,dx\right) = \exp(-x^2)$$

Hence:

$$\int \mu(x)\, q(x)\, dx = \int e^{-x^2}(-x^2)\, dx.$$

$$= \frac{1}{2}\int x\,(-2x) e^{-x^2}\, dx = \frac{1}{2}\int x\,(e^{-x^2})'\, dx$$

Integration by parts.

$$= \frac{1}{2}\left[x e^{-x^2} - \int e^{-x^2}\, dx\right]$$

$$= \frac{x}{2} e^{-x^2} - \frac{1}{2}\int e^{-x^2}\, dx$$

Therefore:

$$v(x) = \frac{1}{\mu(x)}\int \mu(x)\, q(x)\, dx + \frac{C}{\mu(x)}$$

$$= \frac{x}{2} - \frac{1}{2} e^{x^2}\int e^{-x^2}\, dx + C e^{x^2}.$$

Now $u(x) = v(x)^{-1/2}$

$$\boxed{u(x) = \frac{1}{\sqrt{\frac{x}{2} - \frac{1}{2} e^{x^2}\int e^{-x^2}\, dx + C e^{x^2}}}}$$

(3) $x \frac{du}{dx} + (1+x)u = e^{3x}$

$$u(1) = 2e^3$$

Then. $\frac{du}{dx} + \left(1+\frac{1}{x}\right) u = \frac{1}{x} e^{3x}$

Hence: $\mu(x) = \exp\left(\int \left(1+\frac{1}{x}\right) dx\right) = \exp(x + \log x) = xe^x$.

$\int \mu(x) q(x)\, dx = \int x e^x \frac{1}{x} e^{3x} dx = \frac{1}{4} e^{4x}$.

$u(x) = \frac{1}{\mu(x)} \int \mu(x) q(x)\, dx + \frac{C}{\mu(x)} = \frac{1}{xe^x}\left(\frac{1}{4} e^{4x}\right) + \frac{C}{xe^x}$.

$$\boxed{u(x) = \frac{e^{3x}}{4x} + \frac{Ce^{-x}}{x}}$$

Now: $2e^3 = u(1) = \frac{e^3}{4} + Ce^{-1}$

$\Rightarrow 2e^4 = \frac{e^4}{4} + C \Rightarrow C = \left(2 - \frac{1}{4}\right)e^4$

$\Rightarrow C = \frac{7}{4} e^4$. Then: $u(x) = \frac{e^{3x}}{4x} + \frac{7}{4}\frac{e^4 e^{-x}}{x}$.

$$\boxed{u(x) = \frac{1}{4x}\left(e^{3x} + 7e^{4-x}\right)}$$ ✓

④ $$\underbrace{\left(\frac{\sin x}{x} + e^u\right)}_{N(x,u)} \frac{du}{dx} + \underbrace{\frac{u\cos x + e^u}{x}}_{M(x,u)} = 0$$

$$\left.\begin{aligned} \frac{\partial M}{\partial u} &= \frac{\cos x}{x} + \frac{e^u}{x} \\ \frac{\partial N}{\partial x} &= \frac{x\cos x - \sin x}{x^2} \end{aligned}\right\}$$ Not the same
Not exact..

But: $\frac{d\mu}{dx} = \left(\frac{M_u - N_x}{N}\right)\mu = \frac{1}{x}\mu \Rightarrow \frac{d\mu}{dx} = \frac{1}{x}\mu$

$\Rightarrow \boxed{\mu(x) = x}$ is the integrating factor.

Then: $(\sin x + x e^u)\frac{du}{dx} + (u\cos x + e^u) = 0$

is exact. Then:

$\frac{\partial \Psi}{\partial x} = u\cos x + e^u \Longrightarrow \Psi(x,u) = u\sin x + xe^u + g(u)$

$\Downarrow$

$\frac{\partial \Psi}{\partial u} = \sin x + xe^u \qquad \frac{\partial \Psi}{\partial u} = \sin x + xe^u + g'(u)$

Comparing: $g'(u) = 0 \Rightarrow g(u) = \text{const}.$

Then: $\boxed{u\sin x + xe^u = C}$ is the solution.

(5) $T_{amb} = 3°C$ is the weather temperature.

$T(5) = 40°C$ $\quad t =$ time is in minute.

$T(10) = 25°C$

$T(0) = ?$

Diff. Eqn $\quad \frac{dT}{dt} = -k(T - T_{amb})$

Solution: $T(t) = Ce^{-kt} + T_{amb}$.

Hence: $\boxed{T(0) = C + T_{amb}}$

We must compute k and C.

$$T(5) = Ce^{-5k} + T_{amb}$$
$$T(10) = Ce^{-10k} + T_{amb}.$$

$$\Rightarrow \left.\begin{aligned} Ce^{-5k} &= T(5) - T_{amb} \\ Ce^{-10k} &= T(10) - T_{amb} \end{aligned}\right\} \Rightarrow \frac{e^{-5k}}{e^{-10k}} = \frac{T(5) - T_{amb}}{T(10) - T_{amb}}.$$

$$\Rightarrow e^{5k} = \frac{T(5) - T_{amb}}{T(10) - T_{amb}} \Rightarrow \boxed{k = \frac{1}{5} \operatorname{Log}\left|\frac{T(5) - T_{amb}}{T(10) - T_{amb}}\right|}$$

Now: $k = \frac{1}{5}\log\left|\frac{40-3}{25-3}\right| = \frac{1}{5}\operatorname{Log}\left(\frac{37}{22}\right)\frac{1}{\text{min.}}$

$$\Rightarrow 10k = 2\operatorname{Log}\left(\frac{37}{22}\right) \Rightarrow 10k = \operatorname{Log}\left(\frac{37}{22}\right)^2 \Rightarrow e^{10k} = \left(\frac{37}{22}\right)^2$$

Now: $Ce^{-10k} = T(10) - T_{amb} \Rightarrow C = e^{10k}(T(10) - T_{amb})$.

$$\Rightarrow C = \left(\frac{37}{22}\right)^2(25 - 3)$$