UNIVERSIDAD AUTÓNOMA METROPOLITANA - AZCAPOTZALCO ?
ECUACIONES DIFERENCIALES ORDINARIAS
TRIMESTRE: OTOÑO DE 2018.

EXAMEN # 2.
FECHA: **VIERNES 16 DE NOVIEMBRE DE 2018**

Nombre: ANSWER KEY.

Instrucciones:

- El examen consta de **TRES** problemas. Tienen **una** hora con **veinte (20)** minutos para resolverlos.
- Por favor **apaguen sus celulares**. Evíten la pena de quitarles sus exámenes.
- Para recibir puntaje, escriba de forma clara y concisa. Entregue su trabajo limpio y con sus ideas en orden. SIMPLIFIQUE. Muestre sus cuentas. **EXPLIQUE, ARGUMENTE y JUSTIFIQUE** sus respuestas.
- Problema SIN **explicación, desarrollo, justificación o argumento** vale **CERO** puntos.

PROBLEMAS

(1) **(30 puntos.)** Resuelva el problema de valores iniciales

$$3\frac{d^2y}{dt^2} + 9\frac{dy}{dt} - 30y = 0. \qquad y(0) = 0, \qquad y'(0) = 3.$$

(2) **(35 puntos.)** Resuelva la ecuación diferencial:

$$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} - y = t\ln t,$$

(3) **(35 puntos.)** Sin usar el método de factores integrantes, resuelva la ecuación diferencial de **PRIMER** orden:

$$\frac{dy}{dt} - 9y = 2e^{-9t} + 48t^2e^{3t}.$$

Para tener acceso a los puntos extra, deben estar resueltos los problemas anteriores.

(4) **(35 puntos extra.)** De la **LA FORMA** de la solución particular de la ecuación diferencial:

$$\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 18y = t^2e^{3t}\cos(3t).$$

(5) **(35 puntos extra.)** Resuelva la ecuación diferencial:

$$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} - 9y = 6t^3\ln t,$$

(1) The Diff. Eq'n can be written as:

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 10y = 0$$

This is 1) Linear
2) Constant Coeff's
3) Homogeneous

$\Rightarrow y(t) = e^{rt}$

$\Rightarrow r^2 + 3r - 10 = 0$

Then: $(r+5)(r-2) = 0$. $\boxed{r_1 = 2}$ $\boxed{r_2 = -5}$

Solutions: $y_1(t) = e^{2t}$
$y_2(t) = e^{-5t}$.

General solution:

$$y(t) = C_1 y_1(t) + C_2 y_2(t)$$

$$\boxed{y(t) = C_1 e^{2t} + C_2 e^{-5t}}$$

Initial Value Problem:

$y(0) = C_1 + C_2 = 0 \Rightarrow C_2 = -C_1$.

$\dot{y}(0) = 2C_1 - 5C_2 = 2C_1 + 5C_1 = 7C_1 = 3$

$\Rightarrow C_1 = \frac{3}{7} \quad C_2 = -\frac{3}{7}$

Solution: $\boxed{y(t) = \frac{3}{7}\left(e^{2t} - e^{-5t}\right)}$

② Solve the Diff. Eq'n:

$$t^2\ddot{y} + t\dot{y} - y = t\log t.$$

This is a linear, 2nd order, non-homogeneous, non-constant coefficients Diff Eq'n. Use Variation of Parameters.
(Judicious conjecture does not work here, because its non-constant coefficients and forcing term is not of the exp, sine-cosine, polynomial, their products - form).

Step ① Solve the Diff. Eq: (Homogeneous Diff. Eq'n).

$$t^2\ddot{y}_h + t\dot{y}_h - y_h = 0$$

look for solutions of the form: $y_h(t) = t^n$.

$$t^2 n(n-1)t^{n-2} + t n t^{n-1} - t^n = 0.$$

$$n(n-1) + n - 1 = 0 \Rightarrow n^2 - 1 = 0$$

$$\Rightarrow \boxed{n_1 = 1},\ \boxed{n_2 = -1}$$

Two solutions: $y_1(t) = t$

$y_2(t) = t^{-1}$

Then: $y_h(t) = C_1 y_1(t) + C_2 y_2(t)$

$$\boxed{y_h(t) = C_1 t + C_2 t^{-1}}$$

N.

Now: $W[y_1, y_2](t) = \det\begin{pmatrix} t & t^{-1} \\ 1 & -\frac{1}{t^2} \end{pmatrix} = -\frac{2}{t}$, for $t \neq 0$.

The particular solution has the form.

$$y_p(t) = A(t)\,y_1(t) + B(t)\,y_2(t),$$

where

$$A(t) = -\int \frac{y_2(t)\,g(t)}{a(t)\,W[y_1, y_2](t)}\,dt, \quad B(t) = \int \frac{y_1(t)\,g(t)}{a(t)\,W[y_1, y_2](t)}\,dt.$$

where $a(t) = t^2$ We have to solve the integrals.
$g(t) = t \operatorname{Log} t.$

$u = \log t$

$$A(t) = -\int \frac{t^{-1}(t \operatorname{Log} t)}{t^2\left(-\frac{2}{t}\right)}\,dt = \frac{1}{2}\int \frac{\log t}{t}\,dt = \frac{1}{4}(\log t)^2$$

Parts.

$$B(t) = \int \frac{t\,(t \operatorname{Log} t)}{t^2\left(-\frac{2}{t}\right)}\,dt = -\frac{1}{2}\int t \log t\,dt =$$

$$= -\frac{1}{2}\left[\frac{1}{2}t^2 \operatorname{Log} t - \frac{1}{2}\int t^2 \cdot \frac{1}{t}\,dt\right] = -\frac{1}{4}\left[t^2 \log t - \frac{1}{2}t^2\right]$$

$$= -\frac{1}{8}\left(2t^2 \operatorname{Log} t - t^2\right) = -\frac{t^2}{8}\left(2 \operatorname{Log} t - 1\right).$$

Then, the general solution is:

$$y(t) = C_1 y_1(t) + C_2 y_2(t) + y_p(t)$$

$$\boxed{y(t) = C_1 t + \frac{C_2}{t} + \frac{1}{4}(\log t)^2\,t - \frac{t}{8}\left(2 \operatorname{Log} t - 1\right)}$$

(3) This is a linear, constant coeff's, non-homogeneous eq'n.
We can use the Judicious conjecture method.
(There is a variation of parameters method for 1^{st} order Eq's, but did not study here).

Step (1) Solve the homogeneous eq'n: $\dot{y}_h + 9y_h = 0$

Then: $\frac{dy_h}{dt} = -9y_h \Rightarrow \boxed{y_h(t) = e^{-9t}}$.

Also: $\frac{dy_h}{y_h} = -9\,dt \Rightarrow \log|y_h| = -9t \Rightarrow \boxed{y_h(t) = e^{-9t}}$

Step (2) ~~Make~~ Split the problem: into two parts

Part (a) $\dot{y}_p + 9y_p = 2e^{-9t}$.

The guess $\cancel{y_p(t) = Ae^{-9t}}$ does not work,
since the sol'n to the homogeneous question repeats
Then, the right guess is:

$$y_p(t) = Ate^{-9t} \quad \text{(multiply by } t)$$

Then: $$\dot{y}_p(t) = Ae^{-9t} - 9Ate^{-9t} = Ae^{-9t}(1-9t).$$

Hence, substitute into eq'n:

$$Ae^{-9t}(1-\cancel{9t}) + \cancel{9(Ate^{-9t})} = 2e^{-9t}$$

$$\Rightarrow Ae^{-9t} = 2e^{-9t} \Rightarrow \boxed{A=2}$$

Part (b): $\dot{y}_p + 9y_p = 48t^2e^{3t}$.

The judicious conjecture is:

$$y_p(t) = (At^2 + Bt + C)e^{3t}.$$

Solution to the homogeneous eq'n does not repeat. Then, this should work.

$$\dot{y}_p(t) = (2At + B)e^{3t} + 3(At^2 + Bt + C)e^{3t}$$

$$= (3At^2 + (2A + 3B)t + (B + 3C))e^{3t}$$

Substitute into the Diff. Eq'n:

$$(3At^2 + (2A + 3B)t + (B + 3C))\cancel{e^{3t}} + 9(At^2 + Bt + C)\cancel{e^{3t}} =$$

$$= 48t^2\cancel{e^{3t}}.$$

i.e.

$$(3A + 9A)t^2 + (2A + 3B + 9B)t + (B + 3C) + 9C) = 48t^2.$$

System of Eq's

$$12A = 48 \Rightarrow \boxed{A = 4}$$

$$2A + 12B = 0 \Rightarrow B = -\frac{A}{6} \quad \boxed{B = -\frac{2}{3}}$$

$$B + 12C = 0 \Rightarrow C = -\frac{B}{12} \quad \boxed{C = +\frac{1}{18}}$$

Then, the general solution:

$$y(t) = y_h(t) + y_p(t)$$

$$\boxed{y(t) = C_1e^{-9t} + 2te^{-9t} + \left(4t^2 - \frac{2}{3}t + \frac{1}{18}\right)e^{3t}}$$

④ Extra-credit:

We solve first the homogeneous eq'n:

$$\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 18y = t^2 e^{3t}\cos(3t)$$

$$\ddot{y}_h - 6\dot{y}_h + 18y_h = 0 \left\{ \begin{array}{l} 1) \text{ Linear} \\ 2) \text{ Homogeneous} \\ 3) \text{ Const. Coef} \end{array} \right\} y_h(t) = e^{rt}$$

$$r^2 - 6r + 18 = 0 \Rightarrow (r \quad)(r \quad) = 0$$

$$r_{1,2} = \frac{6 \pm \sqrt{36 - 4\cdot 18}}{2} = \frac{6 \pm \sqrt{-36}}{2} = 3 \pm 3i.$$

Then

$$\boxed{y_h(t) = \left(C_1\cos(3t) + C_2\sin(3t)\right)e^{3t}}$$

is the solution ~~to~~ the homogeneous eq'n.

We use the Judicious conjecture:

$$\cancel{y_p(t) = (At^2 + Bt + C)e^{3t}\cos(3t) + (Dt^2 + Et + F)e^{3t}\sin(3t).}$$

does not work because $Ce^{3t}\cos(3t) + Fe^{3t}\sin(3t)$ repeats the sol'n to the homogeneous equations.

Then multiply by t:

$$\boxed{\begin{aligned} y_p(t) = {} & (At^2 + Bt + C)\,t\,e^{3t}\cos(3t) + \\ & + (Dt^2 + Et + F)\,t\,e^{3t}\sin(3t) \end{aligned}}$$

We do not have to solve for A, B, C, D, E, F, since ~~we~~ we were asked just for the form of $y_p(t)$.

⑤ Extra credit. $t^2\ddot{y} + t\dot{y} - 9y = 6t^3 \log t$

This is very similar to ~~eqn~~ problem ②.

Step ① Solve: $t^2\ddot{y}_h + t\dot{y}_h - 9y_h = 0$

Take $y_h(t) = t^n$: $t^2 n(n-1)t^{n-2} + t n t^{n-1} - 9t^n = 0.$

$$\Rightarrow \quad n^2 - n + n - 9 = 0$$

$$\Rightarrow \quad n^2 - 9 = 0 \quad \Rightarrow \quad \boxed{\begin{array}{l} n_1 = 3 \\ n_2 = -3 \end{array}}$$

Solutions $y_1(t) = t^3$, $y_2(t) = t^{-3}$.

Solution to the homogeneous eq'n

$$y_h(t) = C_1 t^3 + \frac{C_2}{t^3}$$

Now: $W[y_1, y_2](t) = \det\begin{pmatrix} t^3 & t^{-3} \\ 3t^2 & -3t^{-4} \end{pmatrix} = -6t^{-1}$, for $t \neq 0$.

The particular solution:

$$y_p(t) = A(t)\, y_1(t) + B(t)\, y_2(t).$$

where:

$$A(t) = -\int \frac{y_2(t)\, g(t)}{a(t)\, W[y_1, y_2](t)}\, dt \; ; \; B(t) = \int \frac{y_1(t)\, g(t)}{a(t)\, W[y_1, y_2](t)}\, dt$$

with $a(t) = t^2$

$g(t) = 6t^3 \log t.$

We have to solve the integrals.

$$A(t) = -\int \frac{t^{-3}\, 6t^3 \log t}{t^2(-6t^{-1})}\, dt = \int \frac{\log t}{t}\, dt = \frac{1}{2}(\log t)^2$$

$u = \log t.$

$$B(t) = \int \frac{t^3\, 6t^3 \log t}{t^2(-6t^{-1})}\, dt = -\int t^5 \log t\, dt$$

By Parts

$$= -\left[\frac{1}{6}t^6 \log t - \frac{1}{6}\int t^6 \cdot \frac{1}{t}\, dt\right] = -\frac{1}{6}\left[t^6 \log t - \frac{t^6}{6}\right]$$

$$= -\frac{t^6}{36}\left[6\log t - 1\right]$$

thus: $y(t) = y_h(t) + y_p(t)$

$$\boxed{y(t) = C_1 t^3 + \frac{C_2}{t^3} + \frac{1}{2}(\log t)^2 t^3 - \frac{t^3}{36}\left[6\log t - 1\right]}$$