UAM - Azcapotzalco Ecuaciones Diferenciales Ordinarias

Examen #3 ANSWER KEY Lunes, Diciembre 3, 2018.

1. The equation of motion is.

$$m\ddot{y} + b\dot{y} + ky = 0.$$

Here, $k = 15\ N/m$, $b = 5\ N \cdot seg/m$.

Hence: $$m\ddot{y} + 5\dot{y} + 15y = 0.$$

To be critically damped, we require.

$$b^2 - 4mk = 0$$

$$5^2 - 4m\cdot(15) = 0.$$

$$25 - 60m = 0 \Rightarrow m = \frac{25}{60}\ kg$$

$$\boxed{m = \frac{5}{12}\ kg}$$

(2) All the units are in the cgs system. So we will keep the cgs system.

The equation of motion is:

$$m\ddot{y} + b\dot{y} + ky = F(t)$$

$m = 400 gr$

$b = 0$ dyn/cm. Since the problem does not mention any friction forces, we assume undamped motion

$k = 1600$ dynes. See computation ahead.

$F(t) = 0$, since there are no external forces:

The initial conditions are:

$y(0) = -10$ cm

$\dot{y}(0) = \sqrt{84}$ cm/seg.

<u>Computation of k:</u> The equilibrium of gravity and Hooke's force:

$$mg = k\,\Delta x \quad \text{Then.} \quad k = \frac{mg}{\Delta x} = \frac{(400)(980)}{245} \ \frac{gr \cdot cm}{cm \cdot seg^2}$$

$k = 1600 \ gr/seg^2 = 1600$ dynes.

<u>Equation of motion</u>

$$400\ddot{y} + 1600y = 0.$$

$$\ddot{y} + 4y = 0$$

The solution is:

$$y(t) = C_1 \cos(2t) + C_2 \sin(2t)$$

The velocity is:

$$\dot{y}(t) = -2C_1 \sin(2t) + 2C_2 \cos(2t).$$

The initial conditions are:

$$\left.\begin{aligned} C_1 &= y(0) = -10 \text{ cm} \\ 2C_2 &= \dot{y}(0) = \sqrt{84}\ \text{cm/seg} \end{aligned}\right\} \Rightarrow \boxed{\begin{aligned} C_1 &= -10 \text{ cm} \\ C_2 &= \sqrt{21} \end{aligned}}$$

Then:

$$y(t) = -10\cos(2t) + \sqrt{21}\sin(2t)$$

However, we are required the amplitude-phase sol'n:

$$y(t) = M\cos(2t - \varphi)$$

Thus:

$$y(t) = M\cos\varphi\cos 2t + M\sin\varphi\sin(2t)$$

Hence

$$\left.\begin{aligned} M\cos\varphi &= -10 \\ M\sin\varphi &= \sqrt{21} \end{aligned}\right\} \left.\begin{aligned} \cos\varphi &< 0 \\ \sin\varphi &> 0 \end{aligned}\right\} \begin{aligned} &\text{Then} \\ &\varphi \in \text{II quadrant} \end{aligned}$$

Then: $\tan\varphi = -\dfrac{\sqrt{21}}{10} \Rightarrow \varphi = -\text{Arctan}\left(\dfrac{\sqrt{21}}{10}\right) + \pi$

$$\varphi \approx -\text{Arctan}(0.46) + \pi \approx -0.43 + \pi \approx 2.71 \text{ rad}.$$

$$\varphi \approx 2.71 \text{ rad} \approx 0.86\,\pi.$$

Now:

$$M = \sqrt{c_1^2 + c_2^2} = \sqrt{10^2 + (\sqrt{21})^2} = \sqrt{121} = 11$$

Then:

$$\boxed{y(t) = 11\cos(2t - 0.86\pi)}$$

We can also start with the solution to

$$\ddot{y} + 4y = 0$$

to be:

$$y(t) = M\cos(2t - \varphi),$$

and then use the initial conditions. We have to compute:

$$\dot{y}(t) = -2M\sin(2t - \varphi)$$

At $t = 0$:

$$M\cos(-\varphi) = y(0)$$
$$-2M\sin(-\varphi) = \dot{y}(0)$$

i.e.

$$\left.\begin{aligned} M\cos\varphi &= -10 \\ 2M\sin\varphi &= \sqrt{84} \end{aligned}\right\} \quad \begin{aligned} M\cos\varphi &= -10 \\ M\sin\varphi &= \sqrt{21} \end{aligned}$$

which are the same conditions as before.

We can also use the MKS system.

$$\boxed{m = \frac{4}{10}\,kg}$$

$$\boxed{b = 0\ N/m}$$

Hooke's law and gravity equilibrium implies:

$$k \cdot \Delta x = mg \quad \Rightarrow \quad k = \frac{mg}{\Delta x} = \frac{\left(\frac{4}{10}\right)(9.8)}{2.45}\ kg/seg^2$$

$$\Rightarrow \quad \boxed{k = \frac{16}{10}\ kg/seg^2}$$

(or N/m)

(Since: $kg = 1000\ gr$;

$$k = \frac{16}{10} \cdot 1000\ gr/seg^2$$

$$= 1600\ gr/seg^2 = 1600\ dyn$$)

Then, the equation of motion is:

$$\frac{4}{10}\ddot{y} + \frac{16}{10}y = 0.$$

i.e. $\boxed{\ddot{y} + 4y = 0}$ Same equation.

The solution is:

$$y(t) = C_1\cos(2t) + C_2\sin(2t)$$

$$\dot{y}(t) = -2C_1\sin 2t + 2C_2\cos(2t).$$

Thei initial conditions: $y(0) = -\frac{1}{10}$ m

$\dot{y}(0) = \frac{\sqrt{84}}{100}$ m/seg.

Then:

$$\left.\begin{aligned} C_1 &= -\frac{1}{10} \\ 2C_2 &= \frac{\sqrt{84}}{100} \end{aligned}\right\} \Rightarrow \begin{aligned} C_1 &= -1/10 \text{ m} \\ C_2 &= \frac{\sqrt{21}}{100} \text{ m} \end{aligned}$$

$$y(t) = -\frac{1}{10}\cos(2t) + \frac{\sqrt{21}}{100}\sin(2t)$$

However, we require.

$$y(t) = M\cos(2t - \varphi).$$

Then: $y(t) = M\cos\varphi\cos 2t + M\sin\varphi\sin(2t)$

$$M\cos\varphi = C_1 = -\frac{1}{10}$$

$$M\sin\varphi = C_2 = \frac{\sqrt{21}}{100}$$

Since $\cos\varphi < 0$, $\sin\varphi > 0 \Rightarrow \varphi \in$ II quadrant.

Then: $\tan\varphi = \frac{M\sin\varphi}{M\cos\varphi} = \frac{\sqrt{21}/100}{-1/10} = \frac{-\sqrt{21}}{10}$

$$\Rightarrow \varphi = \arctan\left(-\frac{\sqrt{21}}{10}\right) + \pi$$

(Since it is on the II quadrant, we add π)

$$\varphi \hat{\approx} -\operatorname{Arctan}(0.46) + \pi \approx -0.43 + \pi \approx 2.71$$

$$\varphi \approx 0.86\pi$$

Finally

$$M = \sqrt{C_1^2 + C_2^2} = \sqrt{\frac{1}{100} + \frac{21}{10000}} = \sqrt{\frac{100 + 21}{10000}}$$

$$= \sqrt{\frac{121}{10000}} = \frac{11}{100}\, m = 11\, cm$$

Same result!

$$\boxed{y(t) = \frac{11}{100}\cos(2t - 0.86\pi)}$$

(3) The Diff. Eq'n is:

$$m\ddot{y} + b\dot{y} + ky = F(t)$$

Here: $m = 1\,gr$

$b = 2\,gr/sag$

$k = 50$

$F(t) = 41\cos(2t)$

Initial conditions.

$y(0) = 23/26$ cm

$\dot{y}(0) = 28 + \frac{2}{13}$ cm/sag

We solve the homogeneous eq'n:

$$\ddot{y}_h + 2\dot{y}_h + 50y_h = 0$$

with solutions $y_h(t) = e^{rt}$, with characteristic eq'n $r^2 + 2r + 50 = 0$, and so $r_{1,2} = \dfrac{-2 \pm \sqrt{4 - 4(50)}}{2}$

$$\boxed{r_{1,2} = -1 \pm 7i}$$

This way:

$$y_h(t) = e^{-t}\left(C_1\cos(7t) + C_2\sin(7t)\right)$$

Since $F(t) = 41\cos(2t)$, look for solutions.

$y_p(t) = A\cos(2t) + B\sin(2t)$. It is ok. It does not repeat solution to homogeneous equation.

Hence $\dot{y}_p(t) = 2B\cos(2t) - 2A\sin(2t)$

$\ddot{y}_p(t) = -4A\cos 2t - 4B\sin(2t)$

The Diff. Eq'n

$$\ddot{y} + 2\dot{y} + 50y = 41\cos(2t)$$

because:

$$(-4A\cos 2t - 4B\sin 2t) + 2(2B\cos 2t - 2A\sin 2t)$$

$$+ 50(A\cos 2t + B\sin 2t) = 41\cos(2t)$$

Putting together coefficients:

$$(-4A + 4B + 50A)\cos(2t) + (-4B - 4A + 50B)\sin(2t)$$

$$= 41\cos(2t)$$

i.e.

$$(46A + 4B)\cos(2t) + (-4A + 46B)\sin(2t) = 41\cos(2t)$$

Then:
$$46A + 4B = 41$$
$$-4A + 46B = 0 \Rightarrow A = \frac{46B}{4} = \frac{23}{2}B.$$

Then.
$$46\left(\frac{23}{2}\right)B + 4B = 41$$
$$((23)^2 + 4)B = 41 \quad B = \frac{41}{23^2+4} = \frac{41}{533} = \frac{1}{13}$$

So,
$$A = \frac{23}{2}\left(\frac{1}{13}\right) = \frac{23}{26} \qquad \boxed{A = \frac{23}{26}} \quad \boxed{B = \frac{1}{13}}$$

Hence, the solution is

$$y(t) = e^{-t}(C_1\cos 7t + C_2\sin 7t) + \frac{23}{26}\cos 2t + \frac{1}{13}\sin(2t)$$

Now, use initial conditions to find C_1, C_2:

$$\dot{y}(t) = -e^{-t}(C_1\cos 7t + C_2 \sin 7t)$$
$$+ 7e^{-t}(-C_1 \sin 7t + C_2 \cos 7t)$$
$$- \frac{23}{13}\sin 2t + \frac{2}{13}\cos 2t$$

Hence: $y(0) = \frac{23}{26}$ m

$\dot{y}(0) = \left(28 + \frac{2}{3}\right)$ m/seg

Hence:

$$\left.\begin{array}{l} C_1 + 0 + \frac{23}{26} + 0 = \frac{23}{26} \\ -C_1 + 0 + 7C_2 + \frac{2}{13} = 28 + \frac{2}{13} \end{array}\right\} \begin{array}{l} \Rightarrow \boxed{C_1 = 0} \\ \Rightarrow 7C_2 = 28 \\ \boxed{C_2 = 4} \end{array}$$

$$\boxed{y(t) = 4e^{-t}\sin(7t) + \frac{23}{26}\cos 2t + \frac{1}{13}\sin 2t}$$