Vienes, Enero 25, 2019

Quiz #1 ANSWER KEY

(1) The definition of the derivative is:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h\to 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})}.$$

$$= \lim_{h\to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h\to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

Then, the derivative is

$$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

(2). The equation of the straigth line is:

$$y - y_0 = m(x - x_0).$$

For the tangent line to the graph of $g(x) = \frac{1}{x}$ is given with $y_0 = g(x_0)$, $m = g'(x_0)$, $x_0 = -2$.

We know: $g'(x) = -\frac{1}{x^2}$ Hence:

$$y_0 = g(-2) = \frac{1}{-2}$$

$$m = g'(-2) = -\frac{1}{(-2)^2} = -\frac{1}{4}$$

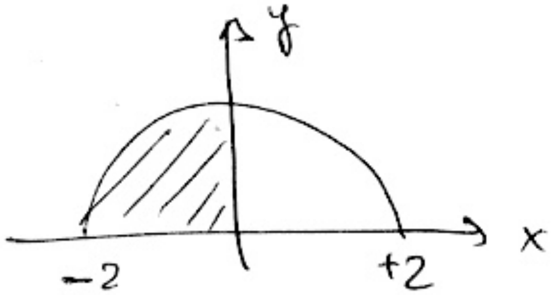
=1=

$$\boxed{y + \frac{1}{2} = -\frac{1}{4}(x+2)}$$

or.

$$\boxed{y = -\frac{1}{4}x - 1}$$

③ The graph of $f(x) = \sqrt{4-x^2}$ is a semi-
of radius 2, center $(0,0)$.



$\int_{-2}^{0} \sqrt{4-x^2}\,dx$ is the area of a quarter of that semi-circle: $\frac{\pi(r^2)}{4} = \frac{\pi(2)^2}{4} = \pi$.

Then: $\boxed{\int_{-2}^{0} \sqrt{4-x^2}\,dx = \pi}$

④ $\int_{-2}^{0} \sqrt{4-x^2}\,dx = \int_{\pi}^{\pi/2} 2\sin\theta(-2\sin\theta)\,d\theta$ (Change of variable $x = 2\cos\theta$)

$= 4\int_{\pi/2}^{\pi} \sin^2\theta\,d\theta = 4\int_{\pi/2}^{\pi} \frac{1-\cos 2\theta}{2}\,d\theta$ $\quad \frac{dx}{d\theta} = -2\sin\theta$

$\sqrt{4-x^2} = 2|\sin\theta|$

$= 2\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_{\pi/2}^{\pi} = 2\left(\pi - \frac{\pi}{2}\right)$ $\quad = 2\sin\theta$

$0 \le \theta \le \pi$

$= \pi$ Same result!

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ANSWER KEY KEY

① The definition of the derivative is:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \lim_{h \to 0} \frac{\left(\frac{x - (x+h)}{(x+h)x}\right)}{h} = \lim_{h \to 0} \frac{-h}{(x+h)x \cdot h}$$

$$= \lim_{h \to 0} \frac{-1}{(x+h)x} = -\frac{1}{x^2}. \quad f'(x).$$

$$\boxed{\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}}$$

② The equation of a ~~tangent~~ line is:

$$y - y_0 = m(x - x_0)$$

For the tangent line we require $y_0 = g(x_0)$

$m = g'(x_0)$

Here, $x_0 = 16$, $g' = \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$, $g(16) = \sqrt{16} = 4$

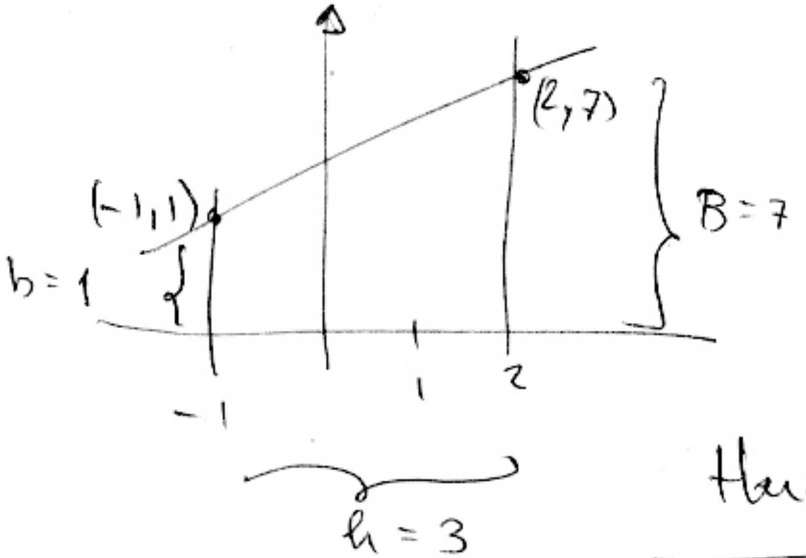
$g'(16) = \frac{1}{2\sqrt{16}} = \frac{1}{2 \cdot 4} = \frac{1}{8}$

Hence $\boxed{y - 4 = \frac{1}{8}(x - 16)}$

Similarly, $y = \frac{1}{8}x - 2 + 4$

$$\boxed{y = \frac{1}{8}x + 2}$$

③ $\int_{-1}^{2}(2x+3)\,dx$ is the area of the trapezoid.



Area of the trapezoid

$$= \left(\frac{b+B}{2}\right)h = \left(\frac{1+7}{2}\right)\cdot 3$$

$$= 4\cdot 3 = 12$$

Hence:

$$\boxed{\int_{-1}^{2}(2x+3)\,dx = 12}$$

(4) $$\int_{-1}^{1} x^2\sqrt{1-x^2}\,dx = \int_{\pi}^{0} \cos^2\theta\,|\sin\theta|(-\sin\theta)\,d\theta$$

$x = \cos\theta$; $\theta \in [0,\pi]$; $\sin\theta \geq 0$ here

$\frac{dx}{d\theta} = -\sin\theta$

$$= \int_0^{\pi} \cos^2\theta \sin^2\theta\,d\theta = \int_0^{\pi} \cos\theta \sin^2\theta \cos\theta\,d\theta$$

$$= \int_0^{\pi} \cos\theta \frac{d}{d\theta}\left(\frac{1}{3}\sin^3\theta\right) d\theta = \frac{1}{3}\cos$$

$$= \frac{1}{3}\cos\theta\sin^3\theta\Big|_0^{\pi} - \frac{1}{3}\int_0^{\pi}(-\sin\theta)\sin^3\theta\,d\theta$$

$$= 0 + \frac{1}{3}\int_0^{\pi}\sin^4\theta\,d\theta = \frac{1}{3}\int_0^{\pi}\sin^2\theta(1-\cos^2\theta)\,d\theta$$

$$= \frac{1}{3}\int_0^{\pi}\sin^2\theta\,d\theta - \frac{1}{3}\int_0^{\pi}\sin^2\theta\cos^2\theta\,d\theta$$

Hence
$$\frac{4}{3}\int_0^{\pi}\sin^2\theta\cos^2\theta\,d\theta = \frac{1}{3}\int_0^{\pi}\frac{1 - \theta}{2}\,d\theta$$
$$= \frac{1}{6}\pi - 0$$
$$\int_0^{\pi}\sin^2\theta\cos^2\theta\,d\theta = \frac{1}{8}\pi$$

Hence $$\boxed{\int_{-1}^{1} x^2\sqrt{1-x^2}\,dx = \int_0^{\pi}\sin^2\theta\cos^2\theta\,d\theta = \frac{1}{8}\pi}$$