UNIVERSIDAD AUTÓNOMA METROPOLITANA - AZCAPOTZALCO
TRIMESTRE: INVIERNO DE 2019.

ECUACIONES DIFERENCIALES ORDINARIAS
EXAMEN # 1.
FECHA: MIÉRCOLES 10 DE JULIO DE 2019.

Nombre: ANSWER KEY.

Instrucciones:

- El examen consta de **CUATRO** problemas de 25 puntos cada uno. Más uno extra de 15 puntos extra.
- Por favor **apaguen sus celulares**. Evíten la pena de quitarles sus exámenes.
- Para recibir puntaje, escriba de forma clara y concisa. Entregue su trabajo limpio y con sus ideas en orden. SIMPLIFIQUE. Muestre sus cuentas. **EXPLIQUE, ARGUMENTE y JUSTIFIQUE** sus respuestas.
- Problema **SIN explicación, desarrollo, justificación o argumento** vale **CERO** puntos.

PROBLEMAS

(1) **(25 puntos.)** ¿Cuál es la FORMA de la solución particular de la siguiente ecuación diferencial?

$$4\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + y = 4t^3e^{t/2}.$$

(2) **(25 puntos.)** Resuelva el problema de valores iniciales:

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 13y = e^{2t}; \qquad y(0) = -1, \qquad \frac{dy}{dt}(0) = 2.$$

(3) **(25 puntos.)** La siguiente ecuación diferencial tiene como solución $y_1(t) = e^t$. Encuentre la otra solución linealmente independiente.

$$t\frac{d^2y}{dt^2} - (t+1)\frac{dy}{dt} + y = 0.$$

(4) **(25 puntos.)** Encuentre la solución general de la ecuación diferencial

$$t^2\frac{d^2y}{dt^2} - 3t\frac{dy}{dt} + 3y = 4t^7.$$

(5) **(15 puntos extra.)** (Condición para resolver este problema: haber resuelto 3 de los anteriores problemas). Resuelva la ecuación diferencial:

$$t\frac{dy}{dt} - y = t^3y^3.$$

UAM - AZCAPOTZALCO. Examen #2. Miércoles, Julio 10, 2019

ECUACIONES DIFERENCIALES ORDINARIAS : ANSWER KEY.

① Step ① Solve the homogeneous equation:

$$4\ddot{y}_h + -4\dot{y}_h + y_h = 0$$

1) Linear
2) Const. Coeff's
3) Homogeneous

$\Big\}\ y_h = e^{rt}$

$$4\left(r^2 - r + \frac{1}{4}\right) = 0.$$

$$4\left(r-\frac{1}{2}\right)^2 = 0$$

$\Rightarrow\ r_1 = r_2 = \frac{1}{2}$: Repeated roots:

$$y_h(t) = (C_1 + C_2 t)\, e^{t/2}$$

Step ② Using the Method of Undetermined Coeff's:

First trial: $Y_1(t) = (At^3 + Bt^2 + Ct + D)\, e^{t/2}$.

It does not work because $(Ct + D)e^{t/2}$ repeats $y_h(t) = (C_1 + C_2 t)e^{t/2}$

Second trial: $Y_2(t) = t(At^3 + Bt^2 + Ct + D)\, e^{t/2}$

It does not work because $Dt\,e^{t/2}$ repeats $C_2 t e^{t/2}$ in $y_h(t)$.

Then: $\boxed{y_p(t) = t^2(At^3 + Bt^2 + Ct + D)\, e^{t/2}}$

and this form should work.

(2) We have to solve the homogeneous equation:

$$\frac{d^2 y_h}{dt^2} - 4\frac{dy_h}{dt} + 13 y_h = 0.$$

1) Linear
2) Const. Coeff'
3) Homogeneous

$y_h(t) = e^{rt} \Rightarrow r^2 - 4r + 13 = 0.$

$$r^2 - 4r + 4 - 4 + 13 = 0$$

$$(r-2)^2 + 9 = 0$$

$$\Rightarrow \boxed{r_{1,2} = 2 \pm 3i}$$

Then: $y_h(t) = e^{2t}\left(C_1\cos(3t) + C_2\sin(3t)\right)$

The particular solution takes the form:

$y_p(t) = Ae^{2t}$, and it works: does not repeat $y_h(t)$.

Thus: $\ddot{y}_p - 4\dot{y}_p + 13 y_p = e^{2t}$

$$2^2 Ae^{2t} - 4\cdot 2Ae^{2t} + 13Ae^{2t} = e^{2t}$$

$$9A = 1 \Rightarrow A = \frac{1}{9}.$$

$\Rightarrow y(t) = e^{2t}\left(C_1\cos(3t) + C_2\sin(3t)\right) + \frac{1}{9}e^{2t}$

is sol'n to the Diff. Eq'n.

Now, $\cancel{\dot{y}(t) = 2y(t) + 3e^{2t}\left(-C_1\sin(3t) + C_2\cos 3t\right) + \frac{2}{9}e^{2t}}$

Now:

$$\dot{y}(t) = 2e^{2t}\left(c_1\cos(3t) + c_2\sin(3t)\right) +$$
$$+ 3e^{2t}\left(-c_1\sin(3t) + c_2\cos(3t)\right) + \frac{2}{9}e^{2t}$$

At $t=0$:
$$y(0) = c_1 + \frac{1}{9}$$
$$\dot{y}(0) = 2c_1 + 3c_2 + \frac{2}{9}$$

$$\left.\begin{aligned} y(0)&=-1 \\ \dot{y}(0)&=2 \end{aligned}\right\} \Rightarrow \quad c_1 + \frac{1}{9} = -1 \Rightarrow \boxed{c_1 = -\frac{10}{9}}$$
$$2\left(-\frac{10}{9}\right) + 3c_2 + \frac{2}{9} = 2 \Rightarrow$$

$$\Rightarrow \quad 3c_2 - 2 = 2 \Rightarrow \boxed{c_2 = \frac{4}{3}}$$

$$\Rightarrow \boxed{y(t) = e^{2t}\left(-\frac{10}{9}\cos(3t) + \frac{4}{3}\sin(3t)\right) + \frac{2}{9}}$$

(3) One solution is: $y_1(t) = e^t$.

The 2nd solution has the form: $y_2(t) = v(t)e^t$.

Find $v(t)$.

$$\dot{y}_2(t) = \dot{v}e^t + ve^t$$
$$\ddot{y}_2(t) = \ddot{v}e^t + 2\dot{v}e^t + ve^t$$

i.e.
$$y_2(t) = ve^t$$
$$\dot{y}_2(t) = (\dot{v} + v)e^t$$
$$\ddot{y}_2(t) = (\ddot{v} + 2\dot{v} + v)e^t$$

$$\Rightarrow 3 =$$

$$t(\ddot{v}+2\dot{v}+v)e^t-(t+1)(\dot{v}+v)e^t+ve^t=0$$

$$t\ddot{v}+\underbrace{2t\dot{v}}+\cancel{tv}-t\dot{v}-\cancel{tv}-\dot{v}-\cancel{v}+\cancel{v}=0$$

$$t\ddot{v}-(t+1)\dot{v}=0$$

$$\Rightarrow V=\dot{v}\Rightarrow t\dot{V}-(t+1)V=0.$$

$$\Rightarrow \frac{\dot{V}}{V}=\left(\frac{t+1}{t}\right)\Rightarrow \frac{d}{dt}(V(t))=1+\frac{1}{t}.$$

$$\Rightarrow \log V(t)=t+\log t=\log(e^t)+\log t.$$

$$=\log(e^t t)=\log(te^t)$$

$$\Rightarrow V(t)=te^t.$$

$$\dot{v}=V=te^t\Rightarrow v(t)=\int te^t\,dt=$$

$$=te^t-\int e^t\,dt=(t-1)e^t$$

$$\Rightarrow \boxed{y_2(t)=(t-1)e^{2t}}$$

$$\Rightarrow \cancel{\boxed{y_2(t)=\left(\int\frac{e^{-t}}{t}dt\right)e^t}}$$

(4) ~~The~~ We have to solve the homogeneous eq'n:

$$t^2 \frac{d^2y}{dt^2} - 3t\frac{dy}{dt} + 3y = 0$$

Assuming $t^m = y_h(t)$:

$$t^2 m(m-1)t^{m-2} - 3t\, m t^{m-1} + 3t^m = 0$$

$$m^2 - 4m + 3 = 0$$

$$\Rightarrow (m-3)(m-1) = 0 \quad \boxed{\begin{matrix} m_1 = 1 \\ m_2 = 3 \end{matrix}} \Rightarrow \begin{matrix} y_1(t) = t \\ y_2(t) = t^3 \end{matrix}$$

$$\boxed{y_h(t) = C_1 t + C_2 t^3}$$

Now, using the method of variation of parameters:

$$y_p(t) = A(t)\, y_1(t) + B(t)\, y_2(t)$$

with
$$A(t) = -\int \frac{y_2(t)\, f(t)}{a(t)\, W[y_1, y_2](t)}\, dt$$

$$B(t) = \int \frac{y_1(t)\, f(t)}{a(t)\, W[y_1, y_2](t)}\, dt$$

Here, $a(t) = t^2$; $f(t) = 4t^7$

We need to compute the wronskian.

$$W[y_1, y_2](t) = \det\begin{pmatrix} t & t^3 \\ 1 & 3t^2 \end{pmatrix} = 3t^3 - t^3 = 2t^3 \neq 0$$

if $t \neq 0$.

Then:

$$A(t) = -\int \frac{t^3 \cdot 4t^7}{t^2 \cdot 2t^3}\, dt = -\int \frac{4t^{10}}{2t^5}\, dt = -2\int t^5\, dt$$

$$= -\frac{2}{6}t^6 = -\frac{1}{3}t^6$$

$$B(t) = \int \frac{t \cdot 4t^7}{t^2 \cdot 2t^3}\, dt = \int \frac{4}{2}\frac{t^8}{t^5}\, dt = 2\int t^3\, dt$$

$$= \frac{2}{4}t^4 = \frac{1}{2}t^4$$

$$\Rightarrow \quad y_p(t) = \left(-\frac{2}{6}t^6\right)t + \left(\frac{2}{4}t^4\right)t^3 = 2\left(\frac{1}{4} - \frac{1}{6}\right)t^7$$

$$= 2\,\frac{2}{24}t^7 = \frac{1}{6}t^7$$

Therefore, the general solution becomes:

$$\boxed{y(t) = C_1 t + C_2 t^3 + \frac{1}{6}t^7}$$

(5) This is a Bernoulli type equation: July 10 2019

$$t\frac{dy}{dt} - y = t^3y^3 \Rightarrow \frac{dy}{dt} - \frac{1}{t}y = t^2y^3$$

$$\Rightarrow y^{-3}\frac{dy}{dt} - \frac{1}{t}y^{-2} = t^2$$

Define $v(t) \equiv y^{-2} \Rightarrow \frac{dv}{dt} = -2y^{-3}\frac{dy}{dt} \Rightarrow -\frac{1}{2}\frac{dv}{dt} = y^{-3}\frac{dy}{dt}$

Hence, the Diff Eq'n becomes:

$$-\frac{1}{2}\frac{dv}{dt} - \frac{1}{t}v = t^2$$

$$\frac{dv}{dt} + \frac{2}{t}v = -2t^2$$

The integrating factor is: $\mu(t) = e^{+\int p(t)dt} = e^{\int \frac{2}{t}dt}$

$$= e^{2\log t} = e^{\log t^2} = t^2.$$

Multiply by $\mu(t) = t^2$:

$$\Rightarrow t^2\frac{dv}{dt} + 2tv = -2t^4$$

$$\frac{d}{dt}(t^2v) = -2t^4 \Rightarrow t^2v(t) = -\frac{2}{5}t^5 + C.$$

$$\Rightarrow v(t) = -\frac{2}{5}t^3 + \frac{C}{t^2}. \quad \text{Now: } y(t) = v^{-2}(t).$$

$$\Rightarrow \boxed{y(t) = \frac{1}{\left(-\frac{2}{5}t^3 + \frac{C}{t^2}\right)^2}} \text{ or } \boxed{y(t) = \frac{25t^2}{(\tilde{C} - 2t^5)^2}}$$