UNIVERSIDAD AUTÓNOMA METROPOLITANA - AZCAPOTZALCO
TRIMESTRE: INVIERNO DE 2019.

**ECUACIONES DIFERENCIALES ORDINARIAS**
**EXAMEN # 3.**
**FECHA DE ENTREGA:**
**JUEVES 18 DE JULIO DE 2019: 4:00 PM, (16:00 H).**

**Nombre:** ANSWER KEY.

**Instrucciones:**

- El examen consta de **TRES** problemas.
- Para recibir puntaje, escriba de forma clara y concisa. Entregue su trabajo limpio y con sus ideas en orden. SIMPLIFIQUE. Muestre sus cuentas. **EXPLIQUE, ARGUMENTE y JUSTIFIQUE** sus respuestas.
- Problema **SIN explicación, desarrollo, justificación o argumento** vale **CERO** puntos.

## PROBLEMAS

(1) **(35 puntos.)** Se tiene un resorte de 24 pulgadas de longitud y se cuelga un cuerpo con masa de 1 slug (32 lb) y se m observa que el resorte se estira 3 pulgadas. Entonces se le cuelga otro cuerpo con masa de 1/2 slug (16 lb) y se deja en equilibrio. Posteriormente se jala la partícula 3 pulgadas hacia arriba dándosele una velocidad de 1 ft/seg hacia abajo. (12 in/seg). Determine la posición de la partícula en todo instante, su amplitud, su fase, su frecuencia angular, su frecuencia natural y su periodo. ((*Hint:* Use el Sistema Inglés: ft, slug, seg). (1 lb = 1 libra). (1 ft = 1 pie). (1 pulgada = 1 in = 1/12 ft).

(2) **(35 puntos.)** Un cuerpo con masa de 0.5 slug (16 lb) se suspende en un resorte con constante de Hooke de 2 lb/ft. La constante de fricción es de $\delta$ lb-seg/ft. Del equilibrio, la partícula se jala hacia abajo 1 ft. y se suelta. Encuentre $\delta$ para que sea un movimiento críticamente amortiguado. Encuentre la posición en todo tiempo. ¿El cuerpo pasa por el origen? Si es así, determine en qué instante lo hace. (*Hint:* Use el Sistema Inglés: ft, slug, seg). (1 lb = 1 libra). (1 ft = 1 pie). (1 pulgada = 1 in = 1/12 ft).

(3) **(30 puntos.)** Considere el sistema masa-resorte libre y amortiguado $my'' + by' + ky = 0$, con condiciones iniciales $y(0) = y_0$, $y'(0) = v_0$.

(a) Considere que es críticamente amortiguado. ¿Qué condiciones se deben tener sobre las condiciones iniciales para que no cruce la posición de equilibrio $y = 0$?

(b) Misma pregunta pero considerando que el sistema es sobre amortiguado.

(*Hint:* Deben ser dos condiciones. Deben estar dadas en forma de desigualdades).

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(1). We use here ft-slug-second (English) system

Hooke's law: $F = k\Delta L$.

$F = 32$ lb (weight, force)

$\Delta L = 3 \text{ in} = \frac{3}{12}$ ft $= \frac{1}{4}$ ft

$k = \frac{F}{\Delta L} = \frac{32}{1/4} \frac{lb}{ft}$

$k = 128$ lb/ft

$\Delta L$ is the increase of the spring

(The original length of the spring is useless).

Now we have a mass of $m = \left(1 + \frac{1}{2}\right)$ slug $= \frac{3}{2}$ slugs.

and $k = 128$ lb/ft

It is assumed no friction : $\underline{b = 0}$.

Then, the equation of motion is:

$$\frac{3}{2}\ddot{y} + 128\, y = 0$$

$$\Rightarrow \ddot{y} = -\frac{256}{3} y \Rightarrow \omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{256}{3}} = \frac{16}{\sqrt{3}} \text{ 1/sec.}$$

is the angular frequency.

The solution to the equation: is:

$$\boxed{y(t) = C_1 \cos\left(\frac{16}{\sqrt{3}} t\right) + C_2 \sin\left(\frac{16}{\sqrt{3}} t\right)}$$

is the position at time $t$ from equilibrium

Now, $\dot{y}(t) = -\frac{16}{\sqrt{3}} C_1 \sin\left(\frac{16}{\sqrt{3}} t\right) + \frac{16}{\sqrt{3}} C_2 \cos\left(\frac{16}{\sqrt{3}} t\right)$.

$\Rightarrow$

At $t=0$ $\quad y_0 = y(0) = C_1$

$\quad v_0 = \dot{y}(0) = \omega C_2.$

$C_1 = y_0 = \frac{1}{4}$ ft

$C_2 = \frac{v_0}{\omega} = \frac{-\sqrt{3}}{16}$ ft

$$\Rightarrow \boxed{y(t) = \frac{1}{4}\cos\left(\frac{16}{\sqrt{3}}t\right) - \frac{\sqrt{3}}{16}\sin\left(\frac{16}{\sqrt{3}}t\right)}$$

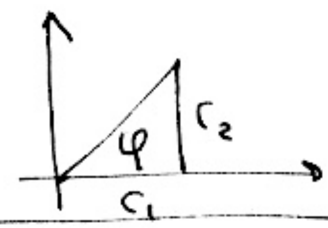
is the position of the particle at time $t$.

with initial conditions considered; position from equilibrium.

The amplitude is given by: $A = \sqrt{C_1^2 + C_2^2} = \sqrt{\frac{1}{4^2} + \left(\frac{-\sqrt{3}}{16}\right)^2}$

$$= \sqrt{\frac{1}{16} + \frac{3}{(16)^2}} = \sqrt{\frac{16+3}{(16)^2}} = \frac{\sqrt{19}}{16} \text{ ft.} \quad \boxed{A = \frac{\sqrt{19}}{16} \text{ ft}}$$

The phase is computed such that




$$\tan\varphi = \frac{C_2}{C_1} = -\frac{\sqrt{3}/16}{1/4} = -\frac{\sqrt{3}}{4}.$$

Since $C_1 > 0$, $\varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \Rightarrow \boxed{\varphi = -\operatorname{Arctan}\left(\frac{\sqrt{3}}{4}\right)}$

Angular frequency: $\boxed{\omega = \sqrt{\frac{k}{m}} = \frac{16}{\sqrt{3}} \ 1/\text{sec}}$

Natural frequency $= \boxed{\nu = \frac{\omega}{2\pi} = \frac{8}{\pi\sqrt{3}} \text{ sec}}$

Period: $\boxed{P = \frac{1}{\nu} = \frac{\pi\sqrt{3}}{8} \text{ sec}}$

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ECUACIONES DIFERENCIALES ORDINARIAS ANSWER KEY

(1) First, we compute Hooke's constant.

We use here ft-slug-second (English system)

Hooke's Law: $F = k\,\Delta L$,

$F = 32\ lb$ (weight, a force)
$\Delta L = 3\ in = \frac{3}{12}\ ft = \frac{1}{4}\ ft.$

$k = \frac{F}{\Delta L} = \frac{32\ lb}{1/4\ ft} = 128\ lb/ft$

$\Delta L$ is the increase of the spring. The length of the spring, 24 in is useless.

If we take away the original particle, and put a new one: i.e.

Now, we have $m = \frac{1}{2}$ slug, $k = 128\ lb/ft$.

$b = 0\ lb\frac{sec}{ft}$, since there is no friction. The eq'n of motion is

$$\frac{1}{2}\ddot{y} + 128 y = 0 \qquad y(0) = 3\ in = \frac{1}{4}\ ft, \quad \dot{y}(0) = -1\frac{ft}{sec}$$

with no external forces and initial conditions.

Then: $\ddot{y} = -256 y \Rightarrow \ddot{y} = -2^8 y.$

$$\Rightarrow \boxed{y(t) = C_1 \cos(2^4 t) + C_2 \sin(2^4 t)}$$

Now: $\dot{y}(t) = -2^4 C_1 \sin(2^4 t) + 2^4 C_2 \cos(2^4 t).$

At $t=0$: $y(0) = C_1 = \frac{1}{4}\ ft = y(0).$

$\dot{y}(0) = 2^4 \cdot C_2 \cdot 1 = -1\ ft/sec.$

$\Rightarrow C_2 = \frac{-1}{16}\ ft.$

Therefore, the solution is:

$$\boxed{y(t) = \frac{1}{4}\cos(16t) - \frac{1}{16}\sin(16t)}$$

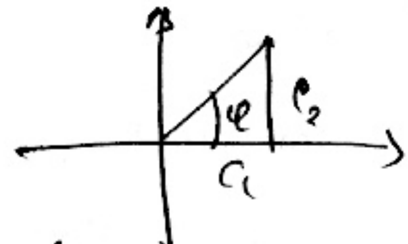
The amplitude is given by:

$$A = \sqrt{c_1^2 + c_2^2} = \sqrt{\frac{1}{4^2} + \frac{1}{(4^2)^2}} = \frac{1}{4^2}\sqrt{4^2+1}$$

$$\boxed{A = \frac{1}{16}\sqrt{17}\ \text{ft}}$$

The ~~fo~~ phase, $\varphi$, is such that $\tan\varphi = \dfrac{-1/16}{1/4} = \dfrac{c_2}{c_1}$

$$\Rightarrow \tan\varphi = -\frac{4}{16} = -\frac{1}{4}$$




Since: $c_1 > 0$, then $\varphi = \text{Arctan}\left(-\frac{1}{4}\right) + 0 \text{ rad}$

$$\boxed{\varphi = -\text{Arctan}\, 1/4}$$

The angular frequency: $\boxed{\omega = 16\ 1/\text{sec}}$

the natural frequency $\dfrac{\omega}{2\pi} = \dfrac{16}{2\pi} = \boxed{\dfrac{8}{\pi}\ \dfrac{1}{\text{sec}} = \nu}$

And the period is: $\boxed{T = \dfrac{1}{\nu} = \dfrac{\pi}{8}\ \text{sec}}$

(2). Here, we have: $m = \frac{1}{2}$ slug, $k = 2$ lb/ft.

$\Rightarrow$ Equation of motion: with initial conditions

$$\frac{1}{2}\ddot{y} + \delta\dot{y} + 2y = 0 \qquad \begin{array}{l} y(0) = -1 \text{ ft} \\ \dot{y}(0) = 0 \text{ ft/sec.} \end{array}$$

We require: $\sqrt{\delta^2 - 4 \cdot mk} = 0$ to be critically damped. Then. $\delta^2 - 4 \cdot \frac{1}{2} \cdot 2 = 0 \Rightarrow \delta^2 = 4$

$$\boxed{\delta = 2 \ \frac{\text{lb}\cdot\text{sec}}{\text{ft}}}$$

The roots of the characteristic eq'n $\frac{1}{2}r^2 + 2r + 2 = 0$

are $r_1 = r_2 = -\frac{\delta}{2m} = -\frac{2}{2 \cdot \frac{1}{2}} = -2$ 1/sec

The solution is:

$$\boxed{y(t) = (C_1 t + C_2)e^{-2t}.}$$

Now: $\dot{y}(t) = C_1 e^{-2t} + (C_1 t + C_2)(-2)e^{-2t}$

$$= \left((C_1 - 2C_2) + (-2)C_1 t\right)e^{-2t}$$

At $t = 0$ $\left.\begin{array}{l} y(0) = -1 \\ \dot{y}(0) = 0 \end{array}\right| \Rightarrow \begin{array}{l} C_2 = y(0) = -1. \\ ((C_1 - 2C_2) + 0)\cdot 1 = 0. \end{array}$

$$\Rightarrow C_1 - 2C_2 = 0 \qquad C_2 = \frac{C_1}{2}$$

$$C_1 = 2C_2 = 2(-1) = -2 \text{ ft/sec.}$$

$C_1 = -2$ ft/sec

$C_2 = -1$ ft

$$\boxed{y(t) = (-2t - 1)e^{-2t}}$$

Since $t > 0$, then $-2t - 2 < 0$ always. Then, it does not passes the origin.

-5-

(3)(a) If it is critically damped,

$$b^2 - 4km = 0. \qquad r_1 = r_2 = \frac{-b}{2m}$$

Then, the roots are repeated $r_1 = r_2 \in \mathbb{R}$ and are real numbers.

$$y(t) = (c_1 t + c_2) e^{-r_1 t}.$$

$$\dot{y}(t) = c_1 e^{-r_1 t} + (c_1 t + c_2)(-r_1) e^{-r_1 t}$$

$$= [c_1 + c_1 t(-r_1) + c_2(-r_1)] e^{-r_1 t}$$

$$= [-r_1 c_1 t + c_1 - r_1 c_2] e^{-r_1 t}$$

$$= -r_1 \left[c_1 t + c_2 - \frac{c_1}{r_1}\right] e^{-r_1 t}$$

At $t=0$: $(0 + c_2) \cdot 1 = y(0) \Rightarrow \boxed{c_2 = y_0}$

$$-r_1\left(c_2 - \frac{c_1}{r_1}\right) = v_0 \Rightarrow -r_1 c_2 + c_1 = v_0$$

$$c_1 = v_0 + r_1 c_2$$

$$\boxed{c_1 = v_0 + r_1 y_0}$$

We require: $y(t) \neq 0$. i.e.

$$c_1 t + c_2 \neq 0$$

(a) $c_1 t + c_2 > 0$ or (b) $c_1 t + c_2 < 0$.

$\Rightarrow$ $c_1 > 0$, $c_2 > 0$ $\boxed{v_0 + r_1 y_0 > 0, \quad y_0 > 0}$ or $c_1 < 0$, $c_2 < 0$ $\boxed{v_0 + r_1 y_0 < 0, \quad y_0 < 0}$

Remark: If: $C_1$ and $C_2$ have opposite signs.

$C_1 > 0$ and $C_2 < 0$ say, then $C_2 = -|C_2|$.

and so: $y(t) = (C_1 t - |C_2|)\, e^{-r_1 t}$.

If $C_1 t - |C_2| = 0$, $y(t) = 0$, and this occur when: $t = \frac{|C_2|}{C_1}$. So, $C_1$ and $C_2$ should have ~~opp~~ the same sign to not to cross the origin.

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(b) If the system is over damped, $r_1 < 0$, $r_2 < 0$. and

$$y(t) = C_1 e^{-|r_1|t} + C_2 e^{-|r_2|t}.$$

If $y(t) = 0$, then: $C_2 e^{-|r_2|t} = -C_1 e^{-|r_1|t}$

$$e^{-(|r_2| - |r_1|)t} = -\frac{C_1}{C_2}.$$

Since $e^{\theta} > 0$, then $-\frac{C_1}{C_2} > 0$ and $C_1, C_2$ have opposite signs.

Then, if $C_1$ and $C_2$ have same signs, the $y(t) \neq 0$

Now, $C_1 + C_2 = y(0) = y_0$

$-|r_1| C_1 - |r_2| C_2 = \dot{y}(0) = v_0$.

$$\begin{pmatrix} 1 & 1 \\ -r_1 & -r_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ v_c \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{-|r_1|-|r_2|} \begin{pmatrix} -r_2 & -1 \\ r_1 & 1 \end{pmatrix} \begin{pmatrix} y_0 \\ v_c \end{pmatrix}$$

$$= \frac{1}{|r_1|+|r_2|} \begin{pmatrix} |r_2| & 1 \\ -|r_1| & -1 \end{pmatrix} \begin{pmatrix} y_0 \\ v_0 \end{pmatrix}$$

$$\boxed{c_1 = \frac{|r_2|\, y_0 + v_c}{|r_1|+|r_2|}}$$

$$\boxed{c_2 = \frac{-|r_1|\, y_0 - v_0}{|r_1|+|r_2|}}$$

Then: either

(a) $c_1 > 0 \quad c_2 > 0 \quad \Rightarrow \quad |r_2|\, y_c + v_0 > 0$

$$-|r_1|\, y_c - v_0 < 0$$

i.e. $\boxed{\begin{aligned} |r_2|\, y_c + v_c > 0 \\ |r_1|\, y_0 + v_0 > 0 \end{aligned}}$

or (b) $c_1 < 0, \ c_2 < 0 \quad \Rightarrow \quad \boxed{\begin{aligned} |r_2|\, y_c + v_0 < 0 \\ |r_1|\, y_c + v_0 < 0 \end{aligned}}$

to avoid crossing $y = 0$.