UNIVERSIDAD AUTÓNOMA METROPOLITANA - AZCAPOTZALCO
ECUACIONES DIFERENCIALES ORDINARIAS
TRIMESTRE: PRIMAVERA DE 2019.

EXAMEN # 1.
FECHA: **VIERNES 11 DE OCTUBRE DE 2019**

Nombre: ANSWER KEY.

- El examen consta de **CINCO** problemas de 20 puntos cada uno.
- Por favor **apaguen sus celulares**. Evíten la pena de quitarles sus exámenes.
- Para recibir puntaje, escriba de forma clara y concisa. Entregue su trabajo limpio y con sus ideas en orden. SIMPLIFIQUE. Muestre sus cuentas. **EXPLIQUE, ARGUMENTE y JUSTIFIQUE** sus respuestas.
- Problema **SIN explicación, desarrollo, justificación o argumento** vale **CERO** puntos.

PROBLEMAS

(1) **(20 puntos.)** La función $\psi(t) = \sin(2t)$, ¿es solución de la ecuación diferencial

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} + 2\cos(2t) + 4y = 0?$$

(2) **(20 puntos.)** Resuelva el problema de valores iniciales:

$$\frac{dy}{dt} = -2 + 2y + \frac{t}{2} - \frac{ty}{2}, \qquad y(0) = -2.$$

(3) **(20 puntos.)** Resuelva el problema de valores iniciales:

$$e^{2t}\frac{dy}{dt} + 3e^{2t}y = e^{-t}\sin(t), \qquad y(0) = 1.$$

(4) **(20 puntos.)** Resuelva la ecuación diferencial.

$$(8y^3 + 1) + 12ty^2\frac{dy}{dt} = 0.$$

(5) **(20 puntos.)** Para la siguiente ecuación diferencial:

$$\frac{dy}{dt} = -y^5 - y^3 + 2y,$$

(a) encuentre los puntos fijos;
(b) bosqueje la línea fase;
(c) de acuerdo al inciso anterior, clasifique los puntos fijos (es decir, geométricamente determine si son fuentes, lavabos o nodos);
(d) clasifique esos mismos puntos fijos por linealización (es decir, analíticamente; es decir, con el criterio de la primera derivada);
(e) bosqueje las soluciones $y(t)$ con diferentes condiciones iniciales.

(6) **(10 puntos extra.)** Escriba el esquema numérico del método de Euler para la siguiente ecuación diferencial con las condiciones iniciales dadas y un paso Δt arbitrario.

$$\frac{dv}{dt} = 4\sin(t^3)y, \qquad v(2) = 1.$$

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ANSWER KEY:

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(1) We have to substitute and check if equality holds

$$\frac{d^2\varphi}{dt^2} - \frac{d\varphi}{dt} + 2\cos(2t) + 4\varphi =$$

$$= \frac{d^2}{dt^2}(\sin 2t) - \frac{d}{dt}(\sin 2t) + 2\cos 2t + 4\sin 2t$$

$$= -\cancel{4\sin 2t} - \cancel{2\cos 2t} + \cancel{2\cos 2t} + \cancel{4\sin 2t} = 0$$

Then, $y = \varphi(t) = \sin(2t)$ is solution.

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(2) This is a separable equation. (It can also be solve by integrating factor.).

$$\frac{dy}{dt} = -2 + 2y + \frac{t}{2} - t\frac{y}{2} = $$

$$= -2(1-y) + \frac{t}{2}(1-y)$$

$$= \left(\frac{t}{2} - 2\right)(1-y)$$

$$\Rightarrow \int \frac{dy}{1-y} = \int \left(\frac{t}{2} - 2\right) dt \Rightarrow -\log|1-y| = t^2 - 2t + C_1$$

$$-\log|1-y| = t^2 - 2t + C_1.$$

$$\Rightarrow 1 - y = e^{-t^2+2t+C_1} = C_2 e^{-t^2+2t}.$$

$$\Rightarrow y(t) = 1 - C_2 e^{-t^2+2t}.$$

$=1=$

At $t=0$, $y(0) = 1 - C_2 e^0$

$$-2 = +1 - C_2$$

$$C_2 = 1+2 = 3$$

$$\Rightarrow \boxed{y(t) = 1 - 3e^{-t^2+2t}}$$

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(3) We solve this eq'n by integrating factor.
We should have the eq'n in standard form, i.e.,
with "1" as a factor of $\dot{y}$:

$$e^{2t}\frac{dy}{dt} + 3e^{2t}y = e^{-t}\sin t$$

$$\frac{dy}{dt} + 3y = e^{-3t}\sin t.$$

The integrating factor satisfies $\frac{d\mu}{dt} = 3\mu$.

$$\Rightarrow \mu(t) = e^{3t}. \text{ Then}$$

$$\mu\frac{dy}{dt} + 3\mu y = \mu e^{-3t}\sin t.$$

$$\Rightarrow e^{3t}\frac{dy}{dt} + 3e^{3t}y = e^{3t}e^{-3t}\sin t$$

$$\frac{d}{dt}\left(e^{3t}y\right) = \sin t$$

$$\Rightarrow e^{3t}y(t) = \int \sin t\,dt + C$$

$$y(t) = -e^{-3t}\cos t + Ce^{-3t}.$$

Using the initial condition: $y(0) = 1$.

$$y(0) = -e^0 \cos(0) + Ce^0$$

$$1 = -1 + C \Rightarrow \underline{C = 2}$$

$$\Rightarrow \boxed{y(t) = -e^{-3t}\cos t + Ce^{-3t}}$$

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(4) We solve this problem by integrating factor $\mu(t,y)$:

$$\underbrace{(8y^3+1)}_{M(t,y)} + \underbrace{12ty^2}_{N(t,y)} \frac{dy}{dt} = 0$$

Check that the derivatives coincide:

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= 24y^2 \\ \frac{\partial N}{\partial t} &= 12y^2 \end{aligned}\right\}$$ They are different. We need an integrating factor:

(a) If $\mu = \mu(t)$; check that $\frac{M_y - N_t}{N}$ is t-independent

$$\frac{M_y - N_t}{N} = \frac{24y^2 - 12y^2}{12ty^2} = \frac{12y^2}{12y^2 t} = \frac{1}{t}$$ It does!

$$\Rightarrow \frac{d\mu}{dt} = \frac{1}{t}\mu \Rightarrow \mu(t) = t.$$

Hence, $(8ty^3 + t) + (12t^2y^2)\frac{dy}{dt} = 0$ is exact:

$$\frac{\partial M_{new}}{\partial y} = 24ty^2 \ \underline{\text{equals}} \ \frac{\partial N_{new}}{\partial t} = 24ty^2$$

Then, there is a function $\Psi(t,y)$, such that.

$$\frac{\partial \Psi}{\partial t} = M_{new}(t,y) = 8ty^3 + t \quad \text{----} (*)$$

$$\frac{\partial \Psi}{\partial y} = N_{new}(t,y) = 12t^2y^2 \quad \text{-----} (**)$$

From $\frac{\partial \Psi}{\partial y} = 12t^2y^2 \Rightarrow \Psi(t,y) = 4t^2y^3 + g(t)$.

$\Rightarrow \frac{\partial \Psi}{\partial t} = 8ty^3 + g'(t)$. Comparing with $(*)$:

$g'(t) = t \Rightarrow g(t) = \frac{t^2}{2} \Rightarrow \Psi(y,t) = 4t^2y^3 + \frac{t^2}{2}$.

and the solution $\Psi(t,y) = C$ becomes

$$\boxed{4t^2y^3 + \frac{t^2}{2} = C}$$

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(5) We have the differential Equation:

$$\frac{dy}{dt} = f(y), \quad \text{with } f(y) = -y^5 - y^3 + 2y$$
$$= -y(y^4 + y^2 - 2)$$
$$= -y(y^2+2)(y^2-1)$$

Then $f(y) = -y(y-1)(y+1)(y^2+2)$

(a) The fixed points satisfy: $f(y) = 0$.

Then $-y(y-1)(y+1)(y^2+1) = 0 \Rightarrow$ $\boxed{\begin{aligned} y_1 &= -1 \\ y_2 &= 0 \\ y_3 &= 1. \end{aligned}}$

are the fixed points.

(b) Evaluate $f(y)$ at points different of fixed points

$$f(-2) = 2(-3)(-1)\left((-2)^2+1\right) = +6(5) = 30 > 0$$

$$\Rightarrow \frac{dy}{dt} > 0 \Rightarrow y(t) \nearrow$$

$$f(-\tfrac{1}{2}) = \tfrac{1}{2}\left(-\tfrac{3}{2}\right)\left(\tfrac{1}{2}\right)\left(\left(-\tfrac{1}{2}\right)^2+2\right) < 0$$

$$\Rightarrow \frac{dy}{dt} < 0 \Rightarrow y(t) \searrow$$

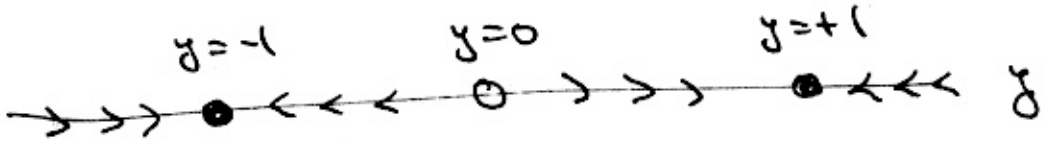
$$f(\tfrac{1}{2}) = -\tfrac{1}{2}\left(-\tfrac{1}{2}\right)\left(\tfrac{1}{2}+1\right)\left(\left(\tfrac{1}{2}\right)^2+1\right) > 0$$

$$\Rightarrow \frac{dy}{dt} > 0 \Rightarrow y(t) \nearrow$$

$$f(2) = -2(1)(3)(2^2+1) < 0$$

$$\frac{dy}{dt} < 0 \Rightarrow y(t) \searrow$$

The phase line is



(c) $y_1 = -1$ is a sink

$y_2 = 0$ is a source

$y_3 = 1$ is a sink

(d) We have to compute: $f'(y) = -5y^4 - 3y^2 + 2$

Now $f'(y_1) = -5 - 3 + 2 = -6 < 0$: $y_1 = -1$, a sink

$= f'(-1) =$

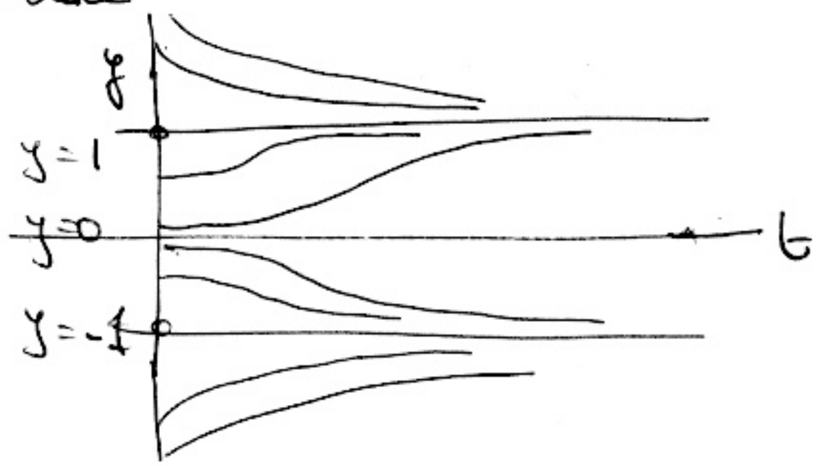
$= 5 =$

$f'(y_2) = f'(0) = 0 + 0 + 2 > 0$: $y_2 = 0$ is a <u>source</u>

$f'(y_3) = f'(1) = -5 - 3 + 2 = -6 < 0$; $y_3 = 1$ is a <u>sink</u>

Coincide with (c).

(e).



⑥ The Diff Eq'n is: $\frac{dv}{dt} = 4\sin(t^3)$.

$v(2) = 1$

Then $\boxed{v_{k+1} = 4\sin(t_k^3)\, v_k \Delta t + v_k}$

for $k = 1, 2, \dots, n$.

and initial condition: $v_0 = 1$.

with. $\boxed{t_k = k\Delta t + t_0}$, and $t_0 = 2$

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PROBLEMAS

(1) **(20 puntos.)** La función $\varphi(t) = 2t$, ¿es solución de la ecuación diferencial

$$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + y = 4t?$$

(2) **(20 puntos.)** Resuelva el problema de valores iniciales:

$$\frac{dy}{dt} = 2 - 4y - t + 2ty, \qquad y(0) = 2.$$

(3) **(20 puntos.)** Resuelva el problema de valores iniciales:

$$\tan(t)\frac{dy}{dt} + y = \tan(t), \qquad y(\pi/2) = 2.$$

(4) **(20 puntos.)** Resuelva la ecuación diferencial.

$$(16ty^3 + 8y^4) + (8ty^3 - 1)\frac{dy}{dt} = 0.$$

(5) **(20 puntos.)** Para la siguiente ecuación diferencial:

$$\frac{dy}{dt} = y^5 - 3y^3 - 12y.$$

(a) encuentre los puntos fijos;
(b) bosqueje la línea fase;
(c) de acuerdo al inciso anterior, clasifique los puntos fijos (es decir, geométricamente determine si son fuentes, lavabos o nodos);
(d) clasifique esos mismos puntos fijos por linealización (es decir, analíticamente; es decir, con el criterio de la primera derivada);
(e) bosqueje las soluciones $y(t)$ con diferentes condiciones iniciales.

(6) **(10 puntos extra.)** Escriba el esquema numérico del método de Euler para la siguiente ecuación diferencial con las condiciones iniciales dadas y un paso Δt arbitrario.

$$\frac{dy}{dt} = t^2\cos(2y^2), \qquad y(1) = 2.$$

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ANSWER KEY.

(1) We should substitute into the Diff. Eqn and should hold:

$$t^2\frac{d^2\varphi}{dt^2}+t\frac{d\varphi}{dt}+\varphi = t^2\left(\frac{d^2}{dt^2}(2t)\right)+t\frac{d}{dt}(2t)+2t$$

$$= t^2\cdot 0 + t\cdot 2t + 2t = 0+2t+2t = 4t.$$

Yes, it is solution

(2) This problem should be also solved by integrating factors. We will do it separating variables:

$$\frac{dy}{dt} = 2-4y-t+2ty = 2(1-2y)-t(1-2y)$$

$$= (2-t)(1-2y) \Rightarrow (t-2)(2y-1)$$

$$\Rightarrow \int\frac{1}{2y-1}\,dy = \int(2-t)\,dt$$

$$\Rightarrow \frac{1}{2}\log|2y-1| = 2t-\frac{t^2}{2}+C_1$$

$$\Rightarrow \log|2y-1| = 4t-t^2+C_2.$$

$$\Rightarrow 2y-1 = C_3e^{-t^2+4t} \Rightarrow y(t)=\frac{1}{2}+C_4e^{-t^2-4t}$$

At $t=0$, $2=\frac{1}{2}+C_4 \Rightarrow C_4 = 3/2$

$$\Rightarrow \boxed{y(t) = \frac{1}{2}+\frac{3}{2}e^{-t^2-4t}}$$

③ We solve this by integrating factors. We should take the Diff. Eq'n in standard form:

$$\tan(t)\frac{dy}{dt} + y = \tan(t) \Rightarrow \boxed{\frac{dy}{dt} + \frac{1}{\tan t}\,dy = 1}$$

The integrating factor should solve $\frac{d\mu}{dt} = \frac{1}{\tan t}\mu$

i.e. $\frac{d\mu}{dt} = \frac{\cos t}{\sin t}\mu \Rightarrow \log\mu = \int \frac{\cos t}{\sin t}\,dt$

$\Rightarrow \log|\mu| = \log|\sin t| \Rightarrow \mu(t) = \sin t$;

Then: $\int \mu(t)g(t)\,dt = \int \sin t \cdot 1\,dt = -\cos t$.

then $y(t) = \frac{1}{\mu(t)}\int \mu(t)\,g(t)\,dt + \frac{C}{\mu(t)}$

becomes $y(t) = \frac{1}{\sin t}(-\cos t) + \frac{C}{\sin t}$.

$$y(t) = -\frac{1}{\tan t} + \frac{C}{\sin t}.$$

At $t = \frac{\pi}{2}$, $\lim_{t\to\frac{\pi}{2}} \tan t = \infty$.

and: i.e. $\left.\frac{1}{\tan t}\right|_{\pi/2} = \left.\frac{\cos t}{\sin t}\right|_{t=\pi/2} = \frac{0}{1} = 0$

Also $\left.\frac{1}{\sin t}\right|_{\pi/2} = \frac{1}{1} = 1$.

thus, $\underset{=2=}{2 = y(\tfrac{\pi}{2})} = 0 + \frac{C}{1} \Rightarrow \underline{C = 2}$

Therefore:

$$\boxed{y(t) = -\frac{\cos t}{\sin t} + \frac{2}{\sin t}}$$

(4) This is solved by generalized integrating factors: $\mu(t,y)$

$$\underbrace{(16ty^3+8y^4)}_{M(t,y)} + \underbrace{(8ty^3-1)}_{N(t,y)}\frac{dy}{dt} = 0.$$

Compute the partial derivatives:

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= 48ty^2 + 32y^3 \\ \frac{\partial N}{\partial t} &= 8y^3 \end{aligned}\right\} \text{Not the same. Require integrating factor } \mu(t,y).$$

(a) Check if $\frac{M_y - N_t}{N}$ depends only on t.

$$\frac{M_y - N_t}{N} = \frac{48ty^2+32y^3-8y^3}{8ty^3-1} = \frac{48ty^2+24y^3}{8ty^3-1}.$$

$$= \frac{(48t+24y)y^2}{8ty^3-1} = \frac{24(2t+y)y^2}{8ty^3-1}. \quad \text{It doesn't work.}$$

(b) Check if $\frac{N_t - M_y}{M}$ depends on y only.

$$\frac{N_t - M_y}{M} = \frac{8y^3 - 48ty^2 - 32y^3}{16ty^3+8y^4} = -\frac{24(2t+y)y^2}{8y^3(2t+y)}$$

$$= -\frac{3}{y}. \quad \text{It works!}$$

Then $\frac{d\mu}{dy} = -\frac{3}{y}\mu \Rightarrow \int \frac{d\mu}{\mu} = \int -\frac{3}{y}\,dy$

$\log|\mu| = -3\log|y| = \log|y^{-3}| \Rightarrow \mu = y^{-3}$.

Therefore, the Diff Eq'n Becomes:

$$\underbrace{(16t + 8y)}_{\tilde{M}(t,y)} + \underbrace{(8t - y^{-3})}_{\tilde{N}(t,y)}\frac{dy}{dt} = 0$$

The derivatives coincide

$$\frac{\partial \tilde{M}}{\partial y} = \frac{\partial}{\partial y}(16t + 8y) = 8$$

$$\frac{\partial \tilde{N}}{\partial t} = \frac{\partial}{\partial t}(8t - y^{-3}) = 8$$

(They coincide
It is an exact
equation.

There is a $\Phi(t,y)$ such that:

$\frac{\partial \Phi}{\partial t} = 16t + 8y \Rightarrow \Phi(t,y) = 8t^2 + 8ty + g(y)$

$\Downarrow$

$\frac{\partial \Phi}{\partial y} = 8t - y^{-3}$ $\quad\quad$ $\frac{\partial \Phi}{\partial y} = 8t + g'(y)$.

Comparing the two $\frac{\partial \Phi}{\partial y}$: $\quad g'(y) = -y^{-3}$.

$\Rightarrow g(y) = \frac{1}{2}y^{-2} \Rightarrow \Phi(t,y) = 8t^2 + 8ty + \frac{1}{2}y^{-2}$.

And the solution is $\boxed{8t^2 + 8ty + \frac{1}{2y^2} = C}$

⑤ We have the Diff. Eq'n $\frac{dy}{dt} = y^5 - 3y^3 - 12y$.

with $f(y) = y^5 - 3y^3 - 12y = y(y^4 - 3y^2 - 12)$

$$= y(y^2 - 4)(y^2 + 3)$$

$$= y(y-2)(y+2)(y^2+3)$$

(a) The fixed points are such that: $f(y) = 0$,

i.e. $\boxed{\begin{array}{l} y_1 = -2 \\ y_2 = 0 \\ y_3 = 2 \end{array}}$

(b) Evaluate f at intermediate points.

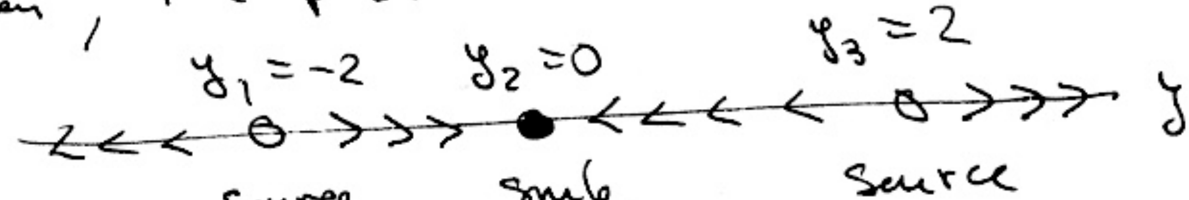
$f(-3) = -3(-5)(-1)(9+3) < 0 \Rightarrow \frac{dy}{dt} < 0$, $y(t) \searrow$

$f(-1) = -1(-3)(1)(1+3) > 0 \Rightarrow \frac{dy}{dt} > 0$, $y(t) \nearrow$

$f(1) = 1(-1)(3)(1+3) < 0 \Rightarrow \frac{dy}{dt} < 0$, $y(t) \searrow$

$f(3) = 3(1)(5)(9+3) > 0 \Rightarrow \frac{dy}{dt} > 0$, $y(t) \nearrow$

Then, the phase line is:



(c) $\boxed{\begin{array}{l} y_1 = -2 \text{ is a } \underline{\text{source}} \\ y_2 = 0 \text{ is a } \underline{\text{sink}} \\ y_3 = 2 \text{ is a } \underline{\text{source}}. \end{array}}$

(d) We need to compute $f'(y)$ and evaluate at the fixed pts.

$$f'(y) = 5y^4 - 9y^2 - 12$$

$$f'(y_1) = f'(-2) = 5\cdot 2^4 - 9\cdot 2^2 - 12 = 80 - 32 - 12$$

$$= 36 > 0$$

$$f'(y_2) = f'(0) = -12 < 0$$

$$f'(y_3) = f'(2) = 5\cdot 2^4 - 9\cdot 2^2 - 12 = 36 > 0$$

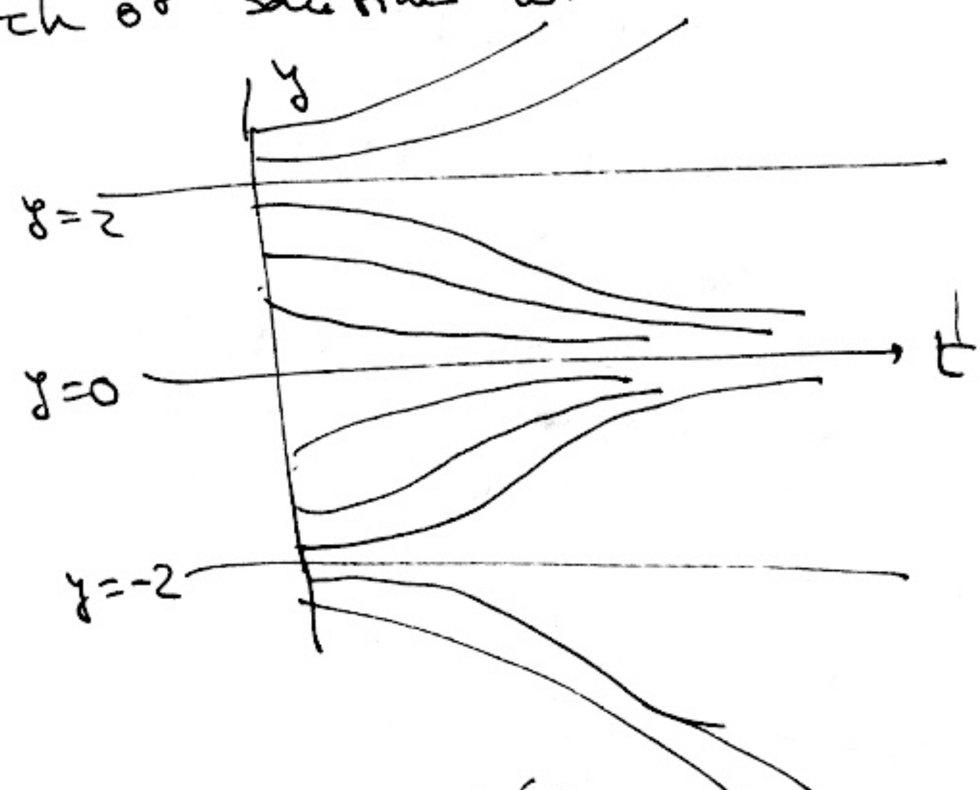
Then, $y_1 = -2$ is a <u>source</u>

$y_2 = 0$ is a <u>sink</u>

$y_3 = 2$ is a <u>source</u>

They coincide.

(e) Sketch of solutions with several initial conditions



(6) The Diff. Eq'n is $\frac{dy}{dt} = t^2 \cos(2y^2)$, $y(1) = 2$.

The Euler's scheme is:

$$\boxed{\begin{array}{c} y_{k+1} = t_k^2 \cos(2y_k^2)\,\Delta t + y_k \\ k = 0, 1, 2, \ldots, n \end{array}}$$

and initial condition $y_0 = 2$.

with $\boxed{t_k = k\Delta t + t_0}$, $t_0 = 1$