UAM - AZCAPOTZALCO | Ecuaciones Diferenciales | V - 25/Oct/2019

Quiz #7. Nombre: ANSWER KEY

(1) Encuentre la familia de trayectorias ortogonales a la familia:
$$y = \frac{1}{2} e^{2kx}$$

(2) Resuelva el PVI.
$$\ddot{y} + 2\dot{y} - 8y = 0 \qquad y(1) = 0, \ \dot{y}(1) = 12$$

Sugerencia: Encuentre soluciones $y(t) = e^{r(t-1)}$

1 es el instante inicial

SOLUTIONS:

(1) We need a Diff. Eq'n for $y(x)$, independent of k:
Write $y = \frac{1}{2} e^{2kx}$ as:
$$\frac{\log(2y)}{x} = 2k$$

Computing its derivative: $-\frac{\log(2y)}{x^2} + \frac{y'}{xy} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{y \log(2y)}{x}$ is the Diff. Eq'n which is k-independent.

Now, if $Y = Y(x)$ are the orthogonal trajectories, the equation $\frac{dy}{dx} \cdot \frac{dY}{dx} = -1$ must hold at the point $(x,y) = (x, Y)$ $\Rightarrow$ $\frac{dY}{dx} = \frac{-1}{\left(\frac{dy}{dx}\right)} = -\frac{x}{y \log(2y)}$

Since $y = Y$ at $(x, y) = (x, Y)$.

$$\frac{dY}{dx} = -\frac{x}{Y \log(2Y)}$$ is the Diff Eq for the orthogonal trajectories.

It is a separable eq'n:

$$\int Y \log(2Y)\, dY = -\int x\, dx.$$

Now

$$u = \log(2Y) \Rightarrow Y = \frac{1}{2}e^{u} \Rightarrow dY = \frac{1}{2}e^{u}\, du$$

This way:

$$\int Y \log(2Y)\, dY = \int \frac{1}{2}e^{u} \cdot u \,\frac{1}{2}e^{2u}\, du = \frac{1}{4}\int u\, e^{2u}\, du$$

$$= \frac{1}{4}\left[u\frac{e^{2u}}{2} - \int \frac{e^{2u}}{2}\, du \right] = \frac{1}{4}\left[\frac{ue^{2u}}{2} - \frac{1}{4}e^{2u} \right]$$

$$= \frac{1}{16}(2u - 1)e^{2u} = \frac{1}{16}\left(2\log 2Y - 1\right)4Y^2$$

$$= \frac{1}{4}\left(2\log(2Y) - 1\right)Y^2$$

Then,

$$\frac{1}{4}\left(2\log 2Y - 1\right)Y^2 = -\frac{x^2}{2} + C_1.$$

$$\boxed{\left(2\log 2Y - 1\right)Y^2 = -2x^2 + C_2.}$$

(2) The Diff Eq'n $\ddot{y} + 2\dot{y} - 8y = 0$ is
1) Linear
2) Const. Coeff's
3) Homogeneous

$$\Rightarrow \quad y(t) = e^{rt}, \quad r = \text{const.}$$

$$r^2 + 2r - 8 = 0 \Rightarrow \begin{array}{l} r_1 = 2 \\ r_2 = -4 \end{array}$$

$$\Rightarrow \quad y(t) = c_1 e^{2(t-1)} + c_2 e^{-4(t-1)}.$$

Hence: $$\dot{y}(t) = 2c_1 e^{2(t-1)} - 4c_2 e^{-4(t-1)}.$$

At $t = 1$:
$$c_1 + c_2 = 0$$
$$2c_1 - 4c_2 = 12.$$

$$\Rightarrow c_2 = -c_1 \Rightarrow 2c_1 + 4c_1 = 12 \Rightarrow \boxed{\begin{array}{l} c_1 = 2 \\ \Rightarrow c_2 = -2 \end{array}}$$

$$\Rightarrow \boxed{y(t) = 2e^{2(t-1)} - 2e^{-4(t-1)}}$$