UNIVERSIDAD AUTÓNOMA METROPOLITANA - AZCAPOTZALCO
ECUACIONES DIFERENCIALES ORDINARIAS
TRIMESTRE: PRIMAVERA DE 2019.

Viernes, 15 de Noviembre de 2019.

EXAMEN # 2.
FECHA: **VIERNES, ~~#~~ DE NOVIEMBRE DE 2019**

Nombre: ANSWER KEY

- El examen consta de **CINCO** problemas de 20 puntos cada uno.
- Por favor **apaguen sus celulares**. Evíten la pena de quitarles sus exámenes.
- Para recibir puntaje, **escriba de forma clara** y concisa. Entregue su trabajo limpio y con sus ideas en orden. SIMPLIFIQUE. Muestre sus cuentas. **EXPLIQUE, ARGUMENTE y JUSTIFIQUE** sus respuestas.
- Problema **SIN explicación, desarrollo, justificación o argumento** vale **CERO** puntos.

PROBLEMAS

(1) **(20 puntos.)** Le sirven un té helado, pero al momento de servirselo, no toma la temperatura. Cinco (5) minutos después, está a 7°C y otros 5 minutos más tarde está a 10°C. ¿A qué temperatura estaba inicialmente? El estado del tiempo dice que la temperatura es de 35°C.

(2) **(20 puntos.)** Resuelva el problema de valores iniciales

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 10y = 0; \qquad y(2) = 3, \qquad \frac{dy}{dt}(2) = 3.$$

(3) **(20 puntos.)** Escriba la forma de la solución particular de la ecuación diferencial

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = t^4e^{2t}.$$

(4) **(20 puntos.)** Resuelva el problema de valores iniciales

$$\frac{d^2y}{dt^2} + 4y = \cot(2t); \qquad y(0) = 1, \qquad \frac{dy}{dt}(0) = 1.$$

Cambiar por $y(\frac{\pi}{4}) = 1$; $\frac{dy}{dt}(\frac{\pi}{4}) = 1$.

(5) **(20 puntos.)** Encuentre la solución general de

$$\frac{d^2y}{dt^2} - \frac{5}{t}\frac{dy}{dt} + \frac{9}{t^2}y = 0$$

(Cero).

UAM-AZCAPOTZALCO | ECUACIONES DIFERENCIALES | Viernes, Noviembre 15, 2019

Examen #2 ANSWER KEY

(1) Assume we are served at time $t=-5$ min. So we want $T(-5)$.

Hence, the temperature at $t=0$ m $T(0)=7°C$

and at $t=5$ m $T(5)=10°C$.

Newton's Cooling law is:

$$\frac{dT}{dt} = -k(T-T_a)$$

where $T_a = 35°C$ is the weather temperature.

The temperature at time t is:

$$T(t) = Ce^{-kt} + T_a$$

or

$$T(t) = (T(0)-T_a)e^{-kt} + T_a$$

Since, $T_a = 35$ and $T(0) = 7°C$,

$$T(t) = -28e^{-kt} + 35 \quad \text{.......} (*)$$

We have to find k, using $T(5)=10$.

$$10 = -28e^{-5k} + 35$$

$$\Rightarrow 28e^{-5k} = 25 \Rightarrow e^{-5k} = \frac{25}{28} \Rightarrow -5k = \log\left(\frac{25}{28}\right)$$

$$\Rightarrow \boxed{k = \frac{1}{5}\log\left(\frac{28}{25}\right)\left(\frac{1}{min}\right)} \approx 0.023 \frac{1}{min}$$

Substitute in eq'n (*).

$$T(t) = -23\, e^{-(0.023)t} + 30$$

At $t = -5$ min.

$$T(-5) = -28\, e^{(0.023)5} + 35$$

$$= -28\, e^{0.1133} + 35$$

$$= -28(1.12) + 35$$

$$\boxed{T(-5) = 3.64\,^{\circ}C}$$

Similarly, since $k = \frac{1}{5} \log\left(\frac{28}{25}\right) = \log\left(\frac{28}{25}\right)^{1/5}$

$$\Rightarrow \quad e^{k} = \left(\frac{28}{25}\right)^{1/5} \Rightarrow e^{kt} = \left(\frac{28}{25}\right)^{t/5}$$

This way, we have that

$$T(t) = -28\, e^{-kt} + 35$$

becomes.

$$T(t) = -28\left(\frac{25}{28}\right)^{t/5} + 35$$

Hence.

$$T(-5) = -28\left(\frac{25}{28}\right)^{-5/5} + 35 = -28\left(\frac{28}{25}\right) + 35$$

$$= -\frac{(28)^2}{25} + 35$$

$$\Rightarrow \boxed{T(-5) = 3.64\,^{\circ}C}$$

Same answer

(2) This is a 1) Linear
2) Constant Coeff's
3) Homogeneous

$\left.\begin{array}{l}\\ \\ \end{array}\right\}$ Diff. Eq'n Then $y(t) = e^{rt}$

Substitute into the Diff. Eq'n. to get $r^2 + 2r + 10r = 0$

i.e. $(r+1)^2 + 9 = 0 \Rightarrow r = -1 \pm 3i$

Then, the general solution is

$$y(t) = e^{-t}(C_1 \cos 3t + C_2 \sin 3t)$$

However, if we use $y(t) = e^{r(t-2)}$,

the solution becomes: $y(t) = e^{(-1+3i)(t-2)} =$

$$= e^{-(t-2)} e^{3i(t-2)} = e^{-(t-2)}(\cos 3(t-2) + i \sin 3(t-2))$$

And so: $\boxed{y(t) = e^{-(t-2)}(C_1 \cos 3(t-2) + C_2 \sin 3(t-2))}$

The derivative is:

$$\dot{y}(t) = -e^{-(t-2)}(C_1 \cos 3(t-2) + C_2 \sin 3(t-2))$$
$$+ e^{-(t-2)}(-3C_1 \sin 3(t-2) + 3C_2 \cos 3(t-2)).$$

Hence, at $t = 2$:

$2 = y(2) = C_1$ $\boxed{C_1 = 2}$

$3 = \frac{dy}{dt}(2) = -C_1 + 3C_2$ $\boxed{C_2 = \frac{3 + C_1}{3} = \frac{5}{3}}$

$$\boxed{y(t) = e^{-(t-2)}\left(2 \cos 3(t-2) + \frac{5}{3} \sin 3(t-2)\right)}$$

(3) Step (1) Solve the homogeneous eq'n:

$$\ddot{y}_h - 4\dot{y}_h + 4y_h = 0$$

1) Linear, 2) Const Coeff's, 3) Homogeneous Diff Eq'n

$y_h(t) = e^{rt}$ $\qquad r^2 - 4r + 4 = 0$

$$(r-2)^2 = 0.$$

$$\Rightarrow \boxed{y_h(t) = (C_1 t + C_2)e^{2t}}$$

Step (2) Propose a particular solution:

$$Y_1(t) = (At^4 + Bt^3 + Ct^2 + \underbrace{Dt + E}_{})e^{2t}$$

This portion repeats the sol'n $y_h(t)$

Multiply by t, then:

$$Y_2(t) = (At^5 + Bt^4 + Ct^3 + Dt^2 + \underline{Et})e^{2t}$$

This repeats sol'n to homogeneous eq'n

Multiply by t again, so we get the particular solution:

$$\boxed{y_p(t) = (At^6 + Bt^5 + Ct^4 + Dt^3 + Et^2)e^{2t}}$$

(1) We have to solve the homogeneous equation.

Step 1 $\frac{d^2 y_h}{dt^2} + 4 y_h = 0$

$$y_h(t) = C_1 \cos(2t) + C_2 \sin(2t)$$

And in Step 2, we use the variation of parameters. We require the Wronskian

$$W[\cos(2t), \sin(2t)] = \det \begin{pmatrix} \cos 2t & \sin(2t) \\ -2\sin 2t & 2\cos(2t) \end{pmatrix} = 2 \neq 0.$$

Then

$$A(t) = -\int \frac{f(t)\, y_2(t)}{a(t) W[y_1,y_2]}\, dt = -\int \frac{\cot(2t)\sin(2t)}{1 \cdot 2}\, dt$$

$$= -\frac{1}{2}\int \frac{\cos(2t)}{\sin(2t)} \sin(2t)\, dt = -\frac{1}{2}\int \cos(2t)\, dt = -\frac{\sin(2t)}{4}$$

$$B(t) = \int \frac{f(t)\, y_1(t)}{a(t) W[y_1,y_2]}\, dt = \int \frac{\cot(2t)\cos(2t)}{1 \cdot 2}\, dt$$

$$= \frac{1}{2}\int \frac{\cos^2(2t)}{\sin 2t}\, dt = \frac{1}{2}\int \frac{1-\sin^2 2t}{\sin(2t)}\, dt$$

$$= \frac{1}{2}\left[\int \frac{1}{\sin(2t)}\, dt - \int \sin(2t)\, dt\right] = \frac{1}{2}\int \frac{dt}{\sin(2t)} + \frac{\cos 2t}{4}.$$

But $\int \frac{1}{\sin 2t}\, dt \underset{y=2t}{=} \frac{1}{2}\int \frac{1}{\sin y}\, dy = \frac{1}{2}\operatorname{Log}|\csc y + \cot y|$

$$= \frac{1}{2}\log|\csc(2t) - \cot(2t)|$$

Hence:

$$B(t) = \frac{1}{4}\operatorname{Log}|\csc(2t) - \cot(2t)| + \frac{\cos(2t)}{4}$$

Then, the particular solution is:

$$y_p(t) = A(t)\, y_1(t) + B(t)\, y_2(t)$$

$$= \cancel{\left(-\frac{\sin(2t)}{4}\right)\cos(2t)} + \frac{1}{4}\left(\log|\csc(2t) - \cot(2t)| + \cancel{\cos 2t}\right)\sin 2t$$

$$= \frac{1}{4}\left(\log|\csc(2t) - \cot(2t)|\right)\sin(2t)$$

Then, the general solution $y(t) = y_h(t) + y_p(t)$ is given by:

$$\boxed{y(t) = c_1\cos(2t) + c_2\sin(2t) + \frac{1}{4}\left(\log|\csc(2t) - \cot(2t)|\right)\sin(2t)}$$

Now, as Laura noticed, at $t=0$, the functions

$$\csc(2t) = \frac{1}{\sin(2t)} \;, \quad \cot(2t) = \frac{\cos(2t)}{\sin(2t)}$$

become "infinite", i.e., they are not defined so we cannot use the initial conditions $y(0)=1$, $\frac{dy}{dt}(0)=1$.

(This should be "obvious" since the Diff. Eq.

$$\ddot{y} + 4y = \cot(4t)$$

does not make sense at $t=0$, because of $\cot(4t)$)

Instead, use the initial conditions:

$$y\left(\tfrac{\pi}{4}\right) = 1, \quad \frac{dy}{dt}\left(\tfrac{\pi}{4}\right) = 1$$

$$1 = y\left(\tfrac{\pi}{4}\right) = C_1 \cos\left(\tfrac{\pi}{2}\right) + C_2 \sin\left(\tfrac{\pi}{2}\right) +$$

$$+ \frac{1}{4}\left(\log\left|\frac{1}{\sin\left(\frac{\pi}{2}\right)} - \frac{\cos\frac{\pi}{2}}{\sin\frac{\pi}{2}}\right|\right)\sin\left(\tfrac{\pi}{2}\right)$$

$$= C_1 \cdot 0 + C_2 \cdot 1 + \frac{1}{4}\left(\log\left|1 - \frac{0}{1}\right|\right) \cdot 1$$

$$= 0 + C_2 + \frac{1}{4}\log|1| = C_2 \Rightarrow C_2 = 1$$

Now:

$$\frac{dy}{dt} = -2C_1 \sin(2t) + 2C_2 \cos(2t) +$$

$$+ \frac{1}{4} \cdot 2 \left(\frac{-\csc(2t)\cot(2t) + \csc^2(2t)}{\csc(2t) - \cot(2t)} \right) \sin(2t)$$

$$+ \frac{1}{4} \left(\log \left| \csc(2t) - \cot(2t) \right| \right) 2 \cdot \cos(2t)$$

At $t = \frac{\pi}{4}$

$$1 = \frac{dy}{dt}\left(\frac{\pi}{4}\right) = -2C_1 + 0$$

$$+ \frac{1}{2} \left(\frac{-1 \cdot 0 + 1}{1 - 0} \right) \cdot 1 \quad + \frac{1}{4} (\log(1)) \cdot 0$$

i.e.

$$1 = -2C_1 + \frac{1}{2} \quad \Rightarrow \quad 2C_1 = \frac{1}{2} - 1 = -\frac{1}{2}$$

$$\boxed{C_1 = -\frac{1}{4}}$$

Therefore, the solution to the Initial Value Problem:

$$\boxed{y(t) = -\frac{1}{4}\cos(2t) + \sin(2t) + \frac{1}{4}\left(\log\left|\csc(2t) - \cot(2t)\right|\right)\sin(2t)}$$

(5) We have the Diff. Eq'n

$$\frac{d^2y}{dt^2} - \frac{5}{t}\frac{dy}{dt} + \frac{9}{t^2}y = 0$$

Multiply by t^2:

1) Linear
2) Homogeneous
3) Non-Constant Coefficients.

$$t^2\frac{d^2y}{dt^2} - 5t\frac{dy}{dt} + 9y = 0$$

$\cancel{y(t) = e^{rt}}$

The order of the derivative coincide with the power of the monomial: Use Euler's method.

Use $y(t) = t^n$

and substitute into the Diff Eq'n

$$t^2 n(n-1)t^{n-2} - 5t\,n t^{n-1} + 9t^n = 0$$

If $t \neq 0$: $\quad n(n-1) - 5n + 9 = 0$

$$n^2 - 6n + 9 = 0 \Rightarrow (n-3)^2 = 0 \quad \boxed{n=3}$$

$\Rightarrow$ We have just one solution:

$$\boxed{y_1(t) = t^n = t^3}$$

We now use the Method of reduction of order:

We propose a second solution.

$$y_2(t) = v(t)\,y_1(t)$$

Substitute into the diff. Eq'n $a(t)\ddot{y} + b(t)\dot{y} + c(t)y = 0$

to get a Diff. Eq'n for $v(t)$:

$$a(t)y_1(t)\frac{d^2v}{dt^2} + \left(2a(t)\dot{y}_1(t) + b(t)y_1(t)\right)\frac{dv}{dt} = 0$$

In our case:

$a(t) = t^2$, $b(t) = -5t$, $y_1(t) = t^3$

$$t^2 \cdot t^3 \frac{d^2v}{dt^2} + \left((2t^2)(3t^2) + (-5t)t^3\right)\frac{dv}{dt} = 0$$

Divide by t^4 and define: $u(t) = \frac{dv}{dt}$. Thus

$$t\frac{du}{dt} + u = 0.$$

which is a 1^{st} order eq'n for $u(t)$ (the order was reduced).

$$\frac{d}{dt}\left(t\,u(t)\right) = 0 \Rightarrow t\,u(t) = C$$

Then: $\frac{dv}{dt} = u = \frac{C}{t} \Rightarrow v(t) = C\log|t|$.
(Set $C=1$)

$\Rightarrow$ $y_2(t) = y_1(t)v(t) = t^3\log|t|$

Then: $$\boxed{y(t) = \left(C_1 t^3 + C_2 t^3 \log|t|\right)}$$

is the general solution to the Diff. Eq'n.