UNIVERSIDAD AUTÓNOMA METROPOLITANA - AZCAPOTZALCO
ECUACIONES DIFERENCIALES ORDINARIAS.
TRIMESTRE: OTOÑO DE 2019: 19-O.

EXAMEN # 1.
FECHA: **VIERNES 31 DE ENERO DE 2020.**

Nombre: Answer Key

Instrucciones:

- El examen consta de **CINCO** problemas de 20 puntos para un total de 100 puntos.
- Tienen **una** hora con **veinticinco (25)** minutos para resolverlos.
- **Apaguen sus celulares**. Evíten la pena de quitarles sus exámenes.
- Para recibir puntaje, consteste correctamente, escriba de forma clara y concisa. Entregue su trabajo limpio y con sus ideas en orden. SIMPLIFIQUE. Muestre sus cuentas. **EXPLIQUE, ARGUMENTE y JUSTIFIQUE** sus respuestas.
- Problema **SIN desarrollo, justificación o argumento** vale **CERO** puntos.

PROBLEMAS

(1) **(20 puntos.)** Resuelva el problema de valores iniciales:

$$\frac{dy}{dt} = t^2y^2 + y^2 + t^2 + 1; \qquad y(0) = 2.$$

(2) **(20 puntos.)** Para la ecuación diferencial,

$$\frac{dy}{dt} = 2y^4 - 8y^2,$$

(a) Encuentre las soluciones de equilibrio.
(b) Bosqueje la línea fase.
(c) Identifique las soluciones de equilibrio como fuentes, lavabos o nodos.
(d) Describa si las soluciones de equilibrio son estables o inestables.
(e) Bosqueje varias gráficas de las soluciones usando la línea fase.
(f) Usando el criterio de la primera derivada (*Teorema de linearización*), determine la estabilidad de las soluciones de equilibrio encontradas en el inciso ~~(2f)~~ (2a).

(3) **(20 puntos.)** La vida-media del plutonio-238 es 87.7 años. Determine el valor de la constate de decaimiento.

(4) **(20 puntos.)** En un caluroso día de verano, se compra un *smoothie* sabor fresa. Después de 5 minutos, el *smoothie* está a 1 °C y después de otros 5 minutos está a 6 °C. ¿A qué temperatura le dieron su *smoothie* inicialmente? La temperatura ambiente es de 30 °C.

(5) Resuelva la siguiente ecuación diferencial usando los siguientes métodos:

$$\frac{dy}{dt} = -2y + 2e^{-2t}.$$

(a) Método de la conjetura sensata.
(b) Método del factor integrante.

ANSWER KEY

① Solving the Initial Value Problem:

$$\frac{dy}{dt} = t^2y^2 + t^2 + y^2 + 1, \quad y(0) = 2.$$

We observe we can factor the right-hand-side out.

$$\frac{dy}{dt} = (t^2+1)(y^2+1).$$

Then, it is separable; By the Theorem of change of variables.

$$\int \frac{dy}{1+y^2} = \int (t^2+1)\,dt.$$

$$\operatorname{Arctan}(y) = \frac{1}{3}t^3 + t + C$$

Using the initial condition: $y(0) = 2$:

$$\boxed{\operatorname{Arctan}(2) = C}$$

thus.

$$y = \tan\left(\frac{1}{3}t^3 + t + \operatorname{Arctan} 2\right).$$

i.e.

$$\boxed{y(t) = \tan\left[\frac{1}{3}t^3 + t + \operatorname{Arctan} 2\right]}$$

(2) For the Diff Eq'n:

$$\frac{dy}{dt} = 2y^4 - 8y^2$$

(a) Find the equilibrium solutions.

The equilibrium solutions are such that: $\frac{dy}{dt} = 0$, ie:

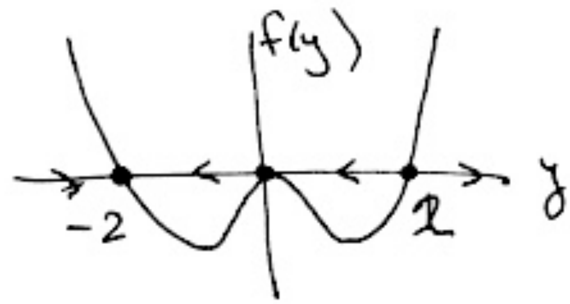
$$2y^4 - 8y^2 = 0 \Rightarrow 2y^2(y^2-4) = 0.$$

$$\Rightarrow \boxed{y_1(t) = -2, \quad y_2(t) = 0, \quad y_3(t) = 2}$$

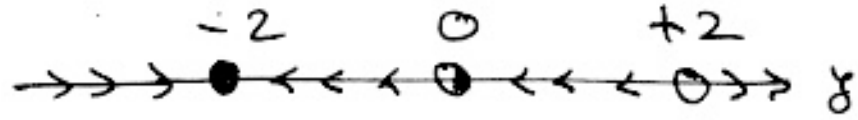
(b) We have that: $f(y) = 2y^2(y^2-4)$ is an even function of y. Then, choosing arbitrary points in the intermediate points in the intervals $(0,2)$, $(2,\infty)$ (defined by the stationary points, we have:

$f(-3) = 90 > 0$ | Because f is even
$f(-1) = -6 < 0$ |
$f(1) = -6 < 0$
$f(3) = 90 > 0$

Then, the graph of $f(y)$ is:



Then, the phase line is:



(c) $y_1(t) = -2$ is a <u>sink</u>. (losabe).

$y_2(t) = 0$ is a <u>node</u>

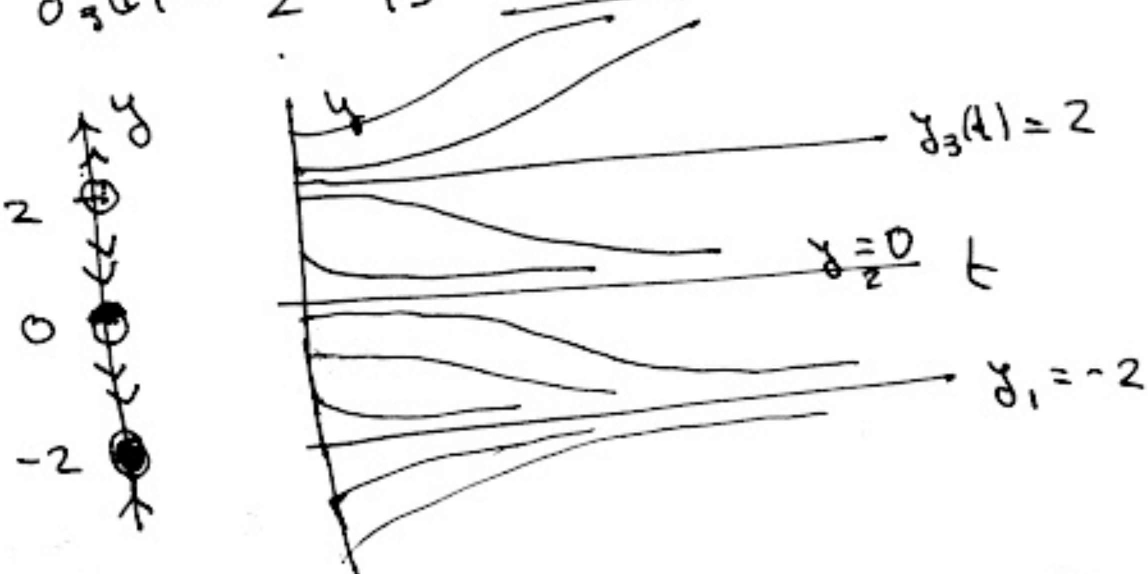
$y_3(t) = 2$ is a <u>source</u> (fuente)

(d) $y_1(t) = -2$ is <u>stable</u>

$y_2(t) = 0$ is <u>unstable</u>. (or semi-stable)

$y_3(t) = 2$ is <u>unstable</u>

(e).



(f). We should compute: $f'(y) = 8y^3 - 16y.$
$= 8y(y^2 - 2).$

$f'(y_1) = f'(-2) = -32 < 0$

$\Rightarrow y_1(t) = -2$ is a <u>sink</u>

and is <u>stable</u>.

$f'(y_2) = f'(0) = 0 \Rightarrow$ we cannot determine the stability of $y_2(t)$

$f'(y_3) = f'(+2) = 32 > 0$

$\Rightarrow y_3(t) = 2$ is a source

and it is unstable.

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③ The half-life of the plutonium -238 is 87.7 years. Find the decay constant:

The mass of the plutonium-238 satisfies the Diff. Eq'n $\frac{dM}{dt} = -kM \Rightarrow M(t) = M(0)e^{-kt}$.

the Half-life, T, is a time such that $M(T) = \frac{M(0)}{2}$. ($T = 87.7$ years).

Then. $M(T) = M(0)e^{-kT}$.

$\frac{M(0)}{2} = M(0)e^{-kT}$. We have to solve for k:

$\frac{1}{2} = e^{-kT}$.

$\Rightarrow e^{kT} = 2 \Rightarrow kT = \log 2 \Rightarrow k = \frac{\log 2}{T}$

Then: $k = \frac{\log 2}{87.7 \text{ years}} \Rightarrow \boxed{k \approx 0.0079 \frac{1}{\text{year}}}$

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④ We have to solve Newton's law of Cooling.

$$\frac{dT}{dt} = -k(T - T_a).$$

with $T_a = 30°C$, and k to be determined. It is separable:

$$\int \frac{1}{T - T_a}\, dT = -k \int dt.$$

$$\Rightarrow \log|T(t) - T_a| = -kt + C_1.$$

$$T(t) - T_a = C_2 e^{-kt}.$$

$$\boxed{T(t) = C_2 e^{-kt} + 30°C}$$

Now, $t = 0_{min}$ is when we first measure the temperature, then, $T(0) = 1°C$ and we have to compute $T(5) = 6°C$ $T(-5)$.

Hence:

$$C_2 e^{-0} + 30°C = 1°C$$

$$\Rightarrow C_2 = -29°C$$

$$\Rightarrow \boxed{T(t) = -29 e^{-kt} + 30°C}$$

With $T(5) = 6°C$, we find k:

$$-29 e^{-5k} + 30°C = 6°C$$

$$\Rightarrow 24°C = 29 e^{-5k} °C \Rightarrow e^{+5k} = \frac{29}{24}$$

$$\Rightarrow \boxed{k = \frac{1}{5} \log \frac{29}{24} \frac{1}{min}} \sim \boxed{k \approx 0.03784 \, 1/min}$$

$$\boxed{T(t) \approx -29e^{-0.03784t} + 30 \ {}^\circ C}$$

At $t = -5$:

$$T(-5) \approx -29e^{(0.03784)5} + 30 \ {}^\circ C$$

$$\approx -29(1.2083) + 30$$

$$\boxed{T(-5) \approx -5.0416\,{}^\circ C}$$

Similarly: $k = \frac{1}{5}\log\left(\frac{29}{24}\right) = \log\left(\frac{29}{24}\right)^{1/5}$

$$\Rightarrow e^{kt} = \left(\frac{29}{24}\right)^{t/5} \Rightarrow \boxed{T(t) = -29\left(\frac{29}{24}\right)^{-t/5} + 30\,{}^\circ C}$$

$$T(0) = -29 + 30\,{}^\circ C = 1\,{}^\circ C$$

$$T(5) = -29\left(\frac{29}{24}\right)^{-1} + 30\,{}^\circ C = -29\left(\frac{24}{29}\right) + 30 = 6\,{}^\circ C$$

$$\Rightarrow T(-5) = -29\left(\frac{29}{24}\right)^{5/5} + 30^\circ = -29\left(\frac{29}{24}\right) + 30$$

$$= -\frac{29^2}{24} + 30 = -5.04\,{}^\circ C$$

$$\boxed{T(-5) = -5.0416\,{}^\circ C}$$

⑤ Solve the Diff Eq. $\frac{dy}{dt} = -2y + 2e^{-2t}$, by

$\boxed{\text{(a) Judicious conjecture:}}$

(i) Solve the homogeneous eq'n:

$$\frac{dy_h}{dt} = -2y_h \Rightarrow y_h(t) = Ce^{-2t}.$$

(ii) Since $\dot{y} + 2y = 2e^{-2t}$, propose: $y_p(t) = Ae^{-2t}$

but this repeats the homogeneous solution. then:

$$y_p(t) = Ate^{-2t}.$$

Substitute onto the Diff Eq'n $\frac{dy}{dt} + 2y = 2e^{-2t}$.

to get:

$$Ae^{-2t} + \cancel{At(-2)e^{-2t}} + \cancel{2(Ate^{-2t})} = 2e^{-2t}$$

$$\Rightarrow Ae^{-2t} = 2e^{-2t} \Rightarrow A = 2.$$

thus the solution is:

$$\boxed{y(t) = Ce^{-2t} + 2te^{-2t}}$$

$\boxed{\text{(b) Factor integrate}}$ For the eq'n in the form:

$$\frac{dy}{dt} + 2y = 2e^{-2t},$$

implies $\mu(t) = e^{\int a(t)dt} = e^{2\int dt} = e^{2t}.$

We then have to integrate:

$$\int \mu(t)\, b(t)\, dt = \int e^{2t}(2e^{-2t})\, dt = \int 2\, dt = 2t.$$

Hence, by the formula

$$y(t) = \frac{1}{\mu(t)} \int \mu(t)\, b(t)\, dt + \frac{C}{\mu(t)}$$

we get:

$$y(t) = \frac{1}{e^{2t}}(2t) + \frac{C}{e^{2t}}$$

i.e. $\boxed{y(t) = 2te^{-2t} + Ce^{-2t}}$

Final answer.