UNIVERSIDAD AUTÓNOMA METROPOLITANA - AZCAPOTZALCO
TRIMESTRE: INVIERNO DE 2020.

CÁLCULO DIFERENCIAL
EXAMEN # 1 (FORMA REMOTA). — AB
FECHA: **SÁBADO 6 DE JUNIO DE 2020: 14:30 HORAS**

Nombre: ANSWER KEY

Instrucciones:

- El examen consta de **SEIS** problemas con diferentes puntajes.
- Tienen **una** hora con **treinta (30)** minutos para resolverlos.
- El examen es **INDIVIDUAL**. Está prohibido recibir ayuda de terceras personas o usar recursos no especificados.
- *Pueden usar sus libros, apuntes y calculadora sencilla. Cite cuando use libros o apuntes. o su* calculadora.
- Para recibir puntaje: Conteste correctamente. Escriba de forma clara y concisa. Entregue su trabajo limpio y con sus ideas en orden. SIMPLIFIQUE y muestre todas sus cuentas. **EXPLIQUE, ARGUMENTE y JUSTIFIQUE** sus respuestas.
- Problema **SIN explicación, desarrollo, justificación o argumento** vale **CERO** puntos.

PROBLEMAS

(0) No olvide elaborar la carátula del examen y anexarla en su escaneado.

(1) **(10 puntos.)** Calcule la ecuación de la recta tangente a la gráfica de la función $g(x) = -\frac{1}{(17-x)^{1/2}}$ en el punto $(1, -1/4)$. Use la definición de derivada.

(2) Calcule:
   (a) **(10 puntos)** la derivada de $f(x) = \sin(\tan^2(\cos 3x))$.
   (b) **(10 puntos)** $(\sin x)^{(139)}$.

(3) **(10 puntos.)** Encuentre la ecuación de la recta ortogonal a la curva $x^3 + y^3 = 16$ en el punto $(2, 2)$.

(4) **(20 puntos.)** Un carro se mueve a lo largo de una autopista y su posición está dada por la función $x(t) = t^4 - 4t^2$. ¿En qué instantes el carro frena?

(5) **(20 puntos.)** Imagine una playa totalmente recta. Un barco va navegando de forma paralela a la playa durante la noche y un faro lo va siguiendo. El barco navega a una distancia de 500 metros de la playa a una velocidad de 200 metros por minuto. El rayo del faro hacia el barco hace un ángulo con la playa. Cuando el barco está a 1000 metros del faro, ¿a qué velocidad angular se mueve el faro?

(6) **(20 puntos.)** En clase se demostró la regla de la cadena. Hay un paso que puede resultar difícil. ¿Cuál es ese paso? ¿En qué consiste dicha dificultad? ¿Cómo la resolvería? Explique.

(*) **FÓRMULAS.**
   (a) $\sin^2\alpha + \cos^2\alpha = 1$.
   (b) $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\cos\beta$
   (c) $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
   (d) $\cos^2\alpha = \frac{1+\cos 2\alpha}{2}$.
   (e) $\sin^2\alpha = \frac{1-\cos 2\alpha}{2}$.

UAM - Azcapotzalco $\boxed{\text{Cálculo Diferencial}}$ Saturday, June 6, 2020

Examen # 1 ANSWER KEY

(1) We have the function $f(x) = -\frac{1}{\sqrt{17-x}}$ and $f(1) = -\frac{1}{4}$, so that the graph passes through $(1, -1/4)$. It remains to find the slope of the tangent line, using the derivative function: $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$. Using this definition:

$$f'(x) = \lim_{h\to 0} \frac{\left(\frac{-1}{\sqrt{17-(x+h)}}\right) - \left(\frac{-1}{\sqrt{17-x}}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\left(\frac{-\sqrt{17-x} + \sqrt{17-(x+h)}}{\sqrt{17-(x+h)}\sqrt{17-x}}\right)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{\sqrt{17-(x+h)} - \sqrt{17-x}}{\sqrt{17-(x+h)}\sqrt{17-x}} \frac{\left(\sqrt{17-(x+h)} + \sqrt{17-x}\right)}{\left(\sqrt{17-(x+h)} + \sqrt{17-x}\right)}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{(17-(x+h)) - (17-x)}{\sqrt{17-(x+h)}\sqrt{17-x}\left(\sqrt{17-(x+h)} + \sqrt{17-x}\right)}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{-h}{\sqrt{17-(x+h)}\sqrt{17-x}\left(\sqrt{17-(x+h)} + \sqrt{17-x}\right)}$$

$$= \lim_{h\to 0} \frac{-1}{(\sqrt{17-(x+h)}\sqrt{17-x}\left(\sqrt{17-(x+h)} + \sqrt{17-x}\right)}$$

Now, we can directly substitute $h=0$; to evaluate limit:

$$= \frac{-1}{\sqrt{17-x}\sqrt{17-x}\left(\sqrt{17-x} + \sqrt{17-x}\right)}$$

$$= \frac{-1}{(17-x)\,2\sqrt{17-x}}$$

$= 1 =$

I.e., the derivative function is:

$$f'(x) = \frac{-1}{2(17-x)^{3/2}}$$

At the value $x=1$, we get the slope of the tangent line:

$$m = f'(1) = \frac{-1}{2(16)^{3/2}} = \frac{-1}{2(4^3)} = \frac{-1}{2^7}.$$

i.e. $m = \frac{1}{128}$. And the equation of the tangent line in point-slope form is

$$y - \left(\frac{-1}{7}\right) = \frac{-1}{128}(x-1)$$

i.e. $$\boxed{y + \frac{1}{4} = \frac{-1}{128}(x-1)}$$

In slope-y-intercept form:

$$y = \frac{-1}{128}x + \frac{1}{128} - \frac{1}{4}. = \frac{1}{128}x + \frac{1-32}{128}$$

$$\boxed{y = \frac{-1}{128}x - \frac{31}{128}}$$

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(2) (a) We use the Chain rule to compute:

$$\frac{d}{dx} f(x) = \frac{d}{dx}\left(\sin\left(\tan^2(\cos 3x)\right)\right)$$

$$= \cos\left(\tan^2(\cos 3x)\right) \frac{d}{dx}\left(\tan^2(\cos 3x)\right) \quad \text{Chain rule}$$

$$= \cos\left(\tan^2(\cos 3x)\right) \cdot 2\tan(\cos 3x) \frac{d}{dx} \tan(\cos 3x) \quad \text{Chain rule}$$

$$= \cos\left(\tan^2(\cos 3x)\right) \cdot 2\tan(\cos 3x) \sec^2(\cos 3x) \cdot \frac{d}{dx}(\cos 3x) \quad \text{Chain rule}$$

$$= \cos\left(\tan^2(\cos 3x)\right) \cdot 2\tan(\cos 3x) \sec^2(\cos 3x)\left(-\sin 3x\right)\frac{d}{dx}(3x) \quad \text{Chain rule}$$

$$= \cos\left(\tan^2(\cos 3x)\right) \cdot 2\tan(\cos 3x) \sec^2(\cos 3x)(-1)\sin(3x) \cdot 3.$$

Rearranging terms.

$$\boxed{\begin{aligned}\frac{df}{dx} &= \frac{d}{dx}\left(\sin\left(\tan^2(\cos 3x)\right)\right) = \\ &= -6\cos\left(\tan^2(\cos(3x))\right)\tan(\cos(3x))\sec^2(\cos(3x))\sin(3x)\end{aligned}}$$

(b) Compute $(\sin x)^{(139)} = \frac{d^{139}}{dx^{139}}(\sin x)$.

Notice that 136 is a multiple of 4. Then.

$$\frac{d^{136}}{dx^{136}} \sin x = \sin x.$$

Hence $\frac{d^{139}}{dx^{139}}(\sin x) = \frac{d^3}{dx^3}\left(\frac{d^{136}}{dx^{136}} \sin x\right) = \frac{d^3}{dx^3}(\sin x) =$

$$= \frac{d^2}{dx^2}(\cos x) = \frac{d}{dx}(-\sin x) = -\cos x$$

$$\Rightarrow \boxed{(\sin x)^{(139)} = -\cos x}$$

(3) We have the function $y = f(x)$ implicitly defined by:

$$x^3 + y^3 = 16.$$

We observe that $(2,2)$ belongs to the curve since

$$2^3 + 2^3 = 8 + 8 = 16 \checkmark$$

Now, compute the derivative on each side of eq'n:

$$3x^2 + 3y^2 y' = 0, \text{ by chain rule.}$$

Then. $$y' = -\frac{x^2}{y^2}.$$

At $(2,2)$, $m_{tan} = y'|_{(2,2)} = -\frac{2^2}{2^2} = -1$,

this is the slope of the tangent line.

The slope of the orthogonal line is:

$$m = -\frac{1}{m_{tan}} = -\frac{1}{-1} = 1$$

Therefore, the equation of the orthogonal line

is: $$y - 2 = 1 \cdot (x - 2)$$

ie $$y - 2 = x - 2$$

$$\Rightarrow \boxed{y = x}$$

(4) The trajectory of the particle is:

$$x(t) = t^4 - 4t^3$$

The derivative is the velocity.

$$\dot{x}(t) = 4t^3 - 12t^2.$$

The acceleration is the second derivative:

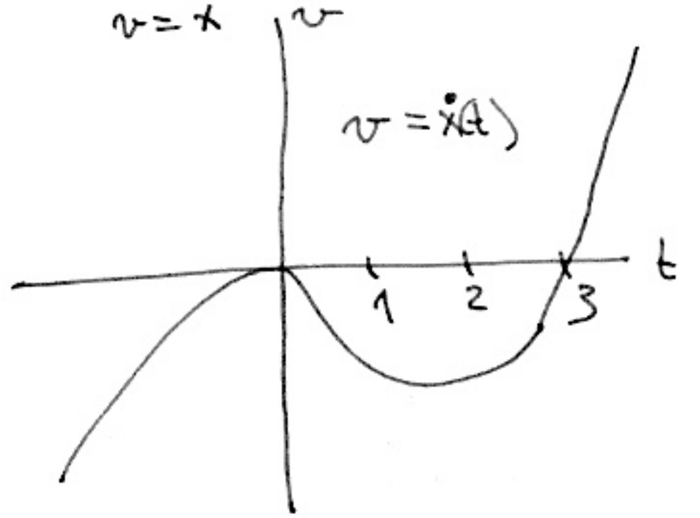
$$\ddot{x}(t) = 12t^2 - 24t.$$

We are asked for the intervals of time when the particle slows down, i.e., when $\dot{x}$ and $\ddot{x}$ have opposite signs.

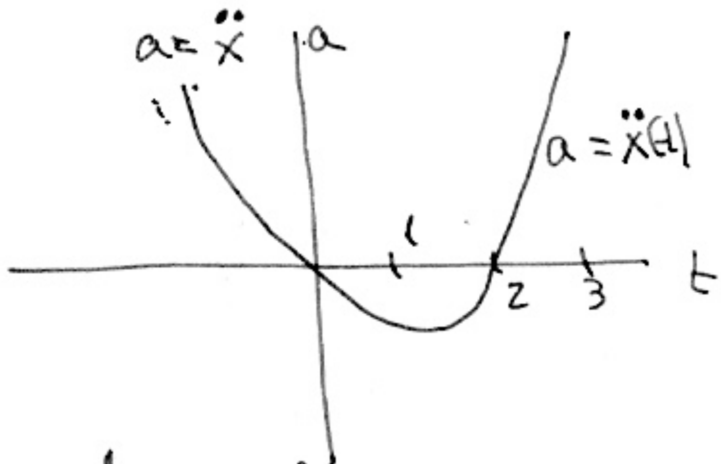
Observe that:

$$\dot{x} = 4t^2(t-3)$$

$$\ddot{x} = 12t(t-2).$$



We can plot these two functions and observe when they have opposite signs.



We have:
$v<0$, $a>0$, on $(-\infty, 0)$.
also:
$v<0$, $a>0$, on $(2,3)$

Then, the particle slows down if

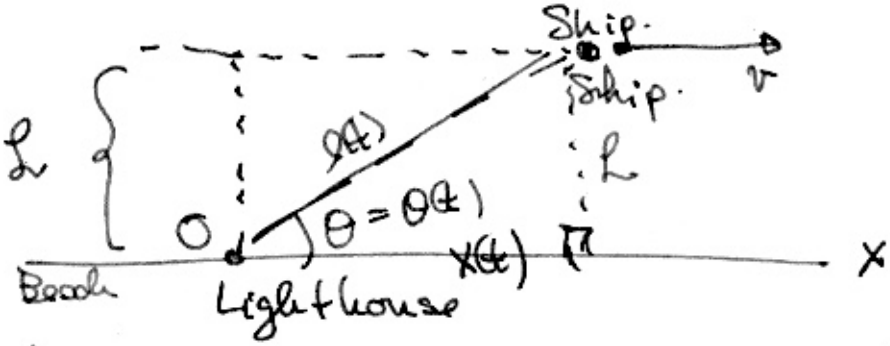
$$\boxed{t \in (-\infty, 0) \cup (2,3)}$$

(In the interval $(0,2) \cup (3,\infty)$, v and a have same signs $\Rightarrow$ speeds up).

(5) Let's do a picture

Here $L = 500\,m$

$v = 200\,m/min$



This are the only constants.

The variables (dependent variables) are: $x(t)$ = projection of ship on beach.

$l = l(t)$ = distance from Lighthouse to ship

$\theta = \theta(t)$ = angle of the ray light and the beach

(time is the independent variable).

Observe that:

$$\tan \theta(t) = \frac{L}{x(t)}$$

since we have a <u>right triangle</u>. Computing the derivatives; by Chain rule:

$$(\sec^2\theta)\,\dot{\theta} = -\frac{L}{x^2}\,\dot{x}$$

$$\text{i.e. } \dot{\theta} = -\frac{L}{x^2}\,\dot{x}\cos^2\theta.$$

But $\cos^2\theta = \left(\frac{x}{l}\right)^2$ Then $\dot{\theta} = -\frac{L}{x^2}\,\dot{x}\left(\frac{x}{l}\right)^2$

i.e. $\dot{\theta} = -\frac{L\,\dot{x}}{l^2}$, since $\dot{x} = v = 200\,m/sec$

and since we want $\dot{\theta}$ when $l = 1000\,m$: $\boxed{\dot{\theta} = -\frac{1}{10}\,\frac{1}{sec}}$

$$\dot{\theta} = -\frac{(500)m\,(200\,m/sec)}{(1000\,m)^2} \Rightarrow \dot{\theta} = \frac{-100{,}000\ m^2/sec}{1{,}000{,}000\ m^2}$$

⑥ We have the composite function.

$$F(x) = (f \circ g)(x) = f(g(x)).$$

If $y = f(u)$, and $u = g(x)$, then $y = f(g(x)) = F(x)$.
and $F'(x) = \frac{dy}{dx}$ is the derivative of F, and it is computed as:

$$\frac{dy}{dx} = \frac{dy}{du}(g(x)) \cdot \frac{du}{dx}(x).$$

The sketch of the proof goes as follows:

A change in x, produces a change in u:

$$x_0 \longmapsto x_0 + \Delta x \implies \Delta u = g(x_0 + \Delta x) - g(x_0).$$

and this change in u, produces a change in y:

$$u_0 \longmapsto u_0 + \Delta u \implies \Delta y = (u_0 + \Delta u) - f(u_0)$$

$(u_0 = g(x_0))$

From this:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \quad \leftarrow \text{Def'n of derivative.}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} \quad \leftarrow \text{Multiply and divide by } \Delta u\text{: \underline{Dangerous} step!}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \quad \leftarrow \text{Prop. of limits}$$

$$= \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \quad \leftarrow \text{Since } \Delta x \to 0 \text{ implies } \Delta u \to 0$$

$$= \frac{dy}{du} \cdot \frac{du}{dx}.$$

That is the dangerous step.

Multiply and divide by Δu: $\frac{\Delta u}{\Delta u}$.

Now, $\frac{\Delta u}{\Delta u} = 1$ and it is okay whenever $\Delta u \neq 0$

If $\Delta u = 0$, we do not have $\frac{\Delta u}{\Delta u} =$ "$\frac{0}{0}$". This

does not make any sense.

We solve this issue as follows:

$\Delta u = 0$ as $x_0 \longmapsto x_0 + \Delta x_0$ means that

$u = g(x)$ is a constant function:

So, we should not use a constant

function, $g(x) \neq$ const, to compose the functions

Or in other words, we require $\frac{dg}{dx} \neq 0$ at $x = x_0$

$\boxed{g'(x_0) \neq 0.}$

I.e., we should use functions where $g'(x_0) \neq 0$.