UNIVERSIDAD AUTÓNOMA METROPOLITANA - AZCAPOTZALCO
TRIMESTRE: INVIERNO DE 2020.

CÁLCULO DIFERENCIAL
EXAMEN # 1 (FORMA REMOTA). — BA
FECHA: **SÁBADO 6 DE JUNIO DE 2020: 14:30 HORAS**

Nombre: ANSWER KEY.

Instrucciones:

- El examen consta de **SEIS** problemas con diferentes puntajes.
- Tienen **una** hora con **treinta (30)** minutos para resolverlos.
- El examen es **INDIVIDUAL**. Está prohibido recibir ayuda de terceras personas o usar recursos no especificados.
- *Pueden usar sus libros, apuntes y calculadora sencilla. Cite cuando use libros o apuntes, o su* calculadora.
- Para recibir puntaje: Conteste correctamente. Escriba de forma clara y concisa. Entregue su trabajo limpio y con sus ideas en orden. SIMPLIFIQUE y muestre todas sus cuentas. **EXPLIQUE, ARGUMENTE y JUSTIFIQUE** sus respuestas.
- Problema **SIN explicación, desarrollo, justificación o argumento** vale CERO puntos.

PROBLEMAS

(0) No olvide elaborar la carátula del examen y anexarla en su escaneado.

(1) **(10 puntos.)** Calcule la ecuación de la recta tangente a la gráfica de la función $g(x) = -\frac{1}{(17-x)^{1/2}}$ en el punto $(1, -1/4)$. Use la definición de derivada.

(2) Calcule:

(a) **(10 puntos)** la derivada de $g(x) = \cos(\tan^2(\sin 5x))$.

(b) **(10 puntos)** $(\cos x)^{(195)}$.

(3) **(10 puntos.)** Encuentre la ecuación de la recta ortogonal a la curva $x^4 + y^4 = 32$ en el punto $(2, -2)$.

(4) **(20 puntos.)** Un automóvil se mueve a lo largo de una carretera y su posición está dada por la función $y(t) = t^4 - 8t^3$. ¿En qué instantes el carro acelera?

(5) **(20 puntos.)** Un pájaro vuela horizontalmente a 20 metros sobre usted a una velocidad de 8 metros por segundo. ¿A qué velocidad angular (respecto a la horizontal) usted va moviendo sus ojos al mirar el pájaro alejarse cuando el pájaro está a 30 metros de usted?

(6) **(20 puntos.)** De un ejemplo de la vida cotidiana en donde aparezca la regla de la cadena (en la cocina, en el carro, en la bicicleta, en las tortillas, en el taller, ...). El ejemplo debe usar solamente aritmética. No debe ser analítico ni algebraico. Use sus propias palabras.

(*) **FÓRMULAS.**

(a) $\sin^2\alpha + \cos^2\alpha = 1$.

(b) $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\cos\beta$

(c) $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

(d) $\cos^2\alpha = \frac{1+\cos 2\alpha}{2}$.

(e) $\sin^2\alpha = \frac{1-\cos 2\alpha}{2}$.

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Examen # 1 ANSWER KEY

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(1) We have the function $f(x) = -\frac{1}{\sqrt{17-x}}$ and $f(1) = -\frac{1}{4}$, so that the graph passes through $(1, -1/4)$. It remains to find the slope of the tangent line, using the derivative function: $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$. Using this definition:

$$f'(x) = \lim_{h\to 0} \frac{\left(\frac{-1}{\sqrt{17-(x+h)}}\right) - \left(\frac{-1}{\sqrt{17-x}}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\left(\frac{-\sqrt{17-x} + \sqrt{17-(x+h)}}{\sqrt{17-(x+h)}\sqrt{17-x}}\right)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{\sqrt{17-(x+h)} - \sqrt{17-x}}{\sqrt{17-(x+h)}\sqrt{17-x}} \frac{\left(\sqrt{17-(x+h)} + \sqrt{17-x}\right)}{\left(\sqrt{17-(x+h)} + \sqrt{17-x}\right)}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{(17-(x+h)) - (17-x)}{\sqrt{17-(x+h)}\sqrt{17-x}\left(\sqrt{17-(x+h)} + \sqrt{17-x}\right)}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{-h}{\sqrt{17-(x+h)}\sqrt{17-x}\left(\sqrt{17-(x+h)} + \sqrt{17-x}\right)}$$

$$= \lim_{h\to 0} \frac{-1}{(\sqrt{17-(x+h)}\sqrt{17-x}\left(\sqrt{17-(x+h)} + \sqrt{17-x}\right)}$$

Now, we can directly substitute $h=0$; to evaluate limit:

$$= \frac{-1}{\sqrt{17-x}\sqrt{17-x}\left(\sqrt{17-x} + \sqrt{17-x}\right)}$$

$$= \frac{-1}{(17-x)\, 2\sqrt{17-x}}$$

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I.e., the derivative function is:

$$f'(x) = \frac{-1}{2(17-x)^{3/2}}$$

At the value $x = 1$, we get the slope of the tangent line:

$$m = f'(1) = \frac{-1}{2(16)^{3/2}} = \frac{-1}{2(4^3)} = \frac{-1}{2^7}.$$

i.e. $m = \frac{1}{128}$. And the equation of the tangent line in point-slope form is

$$y - \left(-\frac{1}{4}\right) = \frac{-1}{128}(x-1)$$

i.e.
$$\boxed{y + \frac{1}{4} = \frac{-1}{128}(x-1)}$$

In slope-y-intercept form:

$$y = \frac{-1}{128}x + \frac{1}{128} - \frac{1}{4} = \frac{1}{128}x + \frac{1-32}{128}$$

$$\boxed{y = \frac{-1}{128}x - \frac{31}{128}}$$

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(2) (a) Compute the derivative of.

$$g(x) = \cos(\tan^2(\sin 5x)).$$

We use the Chain rule:

$$\frac{dg}{dx} = -\sin(\tan^2(\sin 5x))\frac{d}{dx}(\tan^2(\sin(5x)))$$

$$= -\sin(\tan^2(\sin(5x)))\, 2\tan(\sin 5x)\frac{d}{dx}(\tan(\sin(5x)))$$

$$= -\sin(\tan^2(\sin 5x))\, 2\tan(\sin(5x))\sec^2(\sin(5x))\frac{d}{dx}\sin(5x)$$

$$= -\sin(\tan^2(\sin 5x))\, 2\tan(\sin(5x))\sec^2(\sin 5x)\cos(5x)\frac{d}{dx}(5x)$$

$$= -\sin(\tan^2(\sin 5x))\, 2\tan(\sin(5x))\sec^2(\sin 5x)\cos(5x)\cdot 5$$

$$\boxed{\frac{dg}{dx} = -10\sin(\tan^2(\sin(5x)))\tan(\sin(5x))\sec^2(\sin 5x)\cos(5x)}$$

(b) Compute $(\cos x)^{195} = \frac{d^{195}}{dx^{195}}(\cos x)$

Since 192 is a multiple of 4:

$$\frac{d^{192}}{dx^{192}}(\cos x) = \cos x.$$

Then $\frac{d^{195}}{dx^{195}}\cos x = \frac{d^3}{dx^3}\left(\frac{d^{192}}{dx^{192}}\cos x\right) = \frac{d^3}{dx^3}(\cos x)$

$$= \frac{d^2}{dx^2}(-\sin x) = \frac{d}{dx}(-\cos x) = (-1)(-1)\sin x:$$

$$\boxed{\frac{d^{195}}{dx^{195}}(\cos x) = \sin x}$$

(3) The orthogonal line to $x^4+y^4=32$ at $(2,-2)$ is computed as follows.

First, check that $(2,-2)$ is on the curve..

$$x^4+y^4=(2)^4+(-2)^4=16+16=32 \quad \checkmark \text{ It does.}$$

Now, compute implicitly the derivative, by using the Chain rule:

$$\frac{d}{dx}(x^4+y^4)=\frac{d}{dx}(32) \Rightarrow 4x^3+4y^3\frac{dy}{dx}=0.$$

$$\Rightarrow \boxed{\frac{dy}{dx}=-\frac{x^3}{y^3}} \text{ is the implicit derivative.}$$

At $(2,-2)$, the ~~derivative~~ tangent line has slope

$$m_{tan}=\left.\frac{dy}{dx}\right|_{(2,-2)}=-\frac{(2)^3}{(-2)^3}=\frac{2^3}{2^3}=1.$$

Then, the orthogonal line has the slope.

$$m=\frac{-1}{m_{tan}}=\frac{-1}{1}=-1$$

Hence, the equation in point-slope form is:

$$y-(-2)=m(x-(2)) \quad \boxed{y+2=-(x-2)}$$

or in the form slope-y-intercept:

$$y=x-2-2 \quad \boxed{y=-x}$$

4. The function position of the automobile is

$$y(t) = t^4 - 8t^3$$

We compute the velocity: $v = y'(t) = 4t^3 - 24t^2$

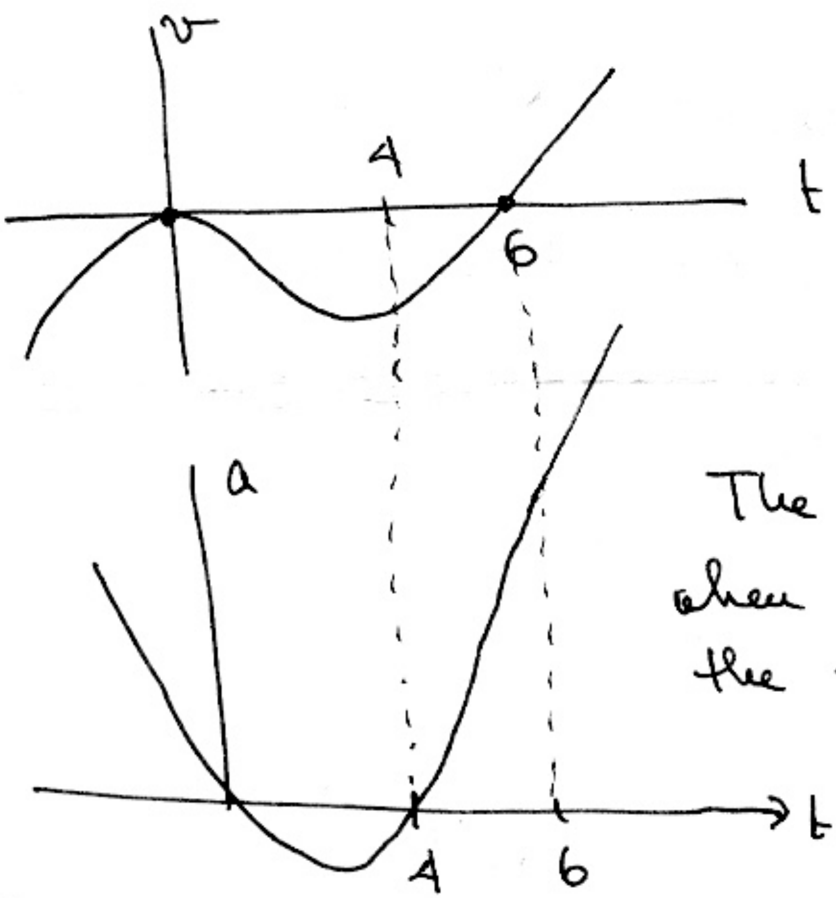
and acceleration: $a = y''(t) = 12t^2 - 48t$.

We can factor the velocity and acceleration.

$$v = 4t^2(t-6)$$

$$a = 12t(t-4)$$

We can plot v and a vs. t:



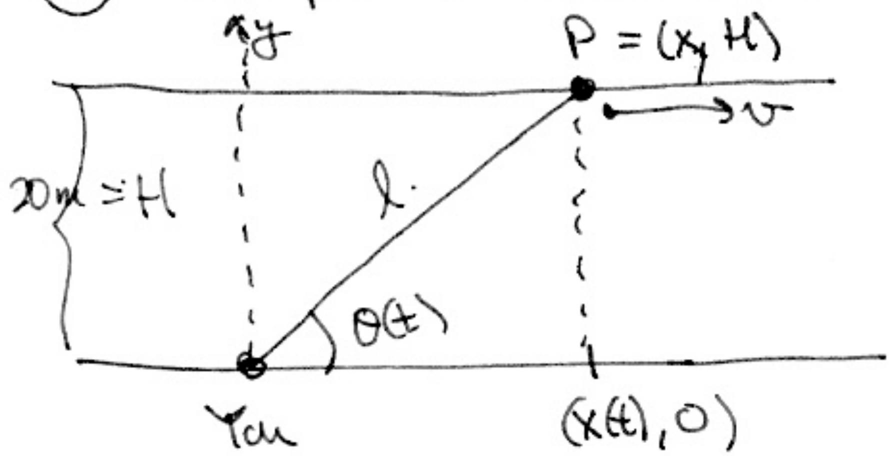
The car speeds up when a and v have the same sign.

Notice that $v > 0$ and $a > 0$ on $(6, \infty)$

and $v < 0$ and $a < 0$ on $(0, 4)$

Then the car speeds up for: $\boxed{t \in (0,4) \cup (6, \infty)}$

(5) We plot a scheme of the situation:



P is the position of the bird, $(x, H) = (x(t), H)$

H is a constant.

$x(t)$ = function of t.

$\theta = \theta(t)$ = angle with respect to the horizontal.

$v = \dot{x}(t)$ is the velocity of bird.

$\theta = \theta(t)$ is also a function of t.

$l = l(t)$ is the distant from you to the bird.

We want $\frac{d\theta}{dt}$, the angular velocity of your eyes.

Then, $\tan(\theta(t)) = \frac{H}{x(t)}$ Computing the derivative on both side.

$$\sec^2(\theta(t))\frac{d\theta}{dt} = -\frac{H}{x^2}\frac{dx}{dt}$$

Since $\sec^2\theta = \frac{1}{\cos^2\theta} = \left(\frac{l}{x}\right)^2$ then we have:

$$\left(\frac{l}{x}\right)^2\frac{d\theta}{dt} = -\frac{H}{\cancel{x^2}}\frac{dx}{dt} \Rightarrow \boxed{\frac{d\theta}{dt} = -\frac{H}{l^2}\frac{dx}{dt}}$$

Related rates

When $l = 30$ m, the bird is 30 m. away, and since $H = 20$ m, $\frac{dx}{dt} = v = 6$ m/sec, we have

$$\frac{d\theta}{dt} = -\frac{20m}{30^2 m^2}\cdot\frac{6m}{sec} = -\frac{120}{900}\frac{1}{sec} = -\frac{4}{30}\frac{1}{sec}$$

$$\frac{d\theta}{dt} \approx -0.1333\ 1/sec \qquad \boxed{\frac{d\theta}{dt} = -\frac{2}{15}\frac{1}{sec}}$$

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(6). Examples like this one are very often. Assume you are a chef and you have to prepare cakes for a party.

You know that for 15 persons you need one cake. and for each cake you need 30gr of sugar.

If the party is of 120 persons, whow much sugar you require? You should use the chain rule:

$$\underbrace{\underbrace{120 \cdot \frac{1}{15}}_{\text{Amount of cakes}} \cdot 30}_{\text{Amount of sugar.}} = 8 \cdot 30 = 240 \text{ grams.}$$

Here we have the chain rule twice.

$$(120 \text{ persons}) \cdot \left(\frac{1}{15} \frac{\text{cake}}{\text{person}} \right) \cdot \left(30 \frac{\text{grams}}{\text{cake}} \right)$$

$$\boxed{= \frac{120}{15} \cdot 30 \text{ grams} = 240 \text{ grams}}$$